Type II compactifications on manifolds with SU(2) × SU(2) structure

Type II compactifications on manifolds with $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure

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ABSTRACT: We study compactifications of type II theories on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure manifolds to six, five and four spacetime dimensions. We use the framework of generalized geometry to describe the NS-NS sector of such compactifications and derive the structure of their moduli spaces. We show that in contrast to $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure compactifications, there is no dynamical $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure interpolating between an $\mathrm{SU}(2)$ structure and an identity structure. Furthermore, we formulate type II compactifications on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures in the context of exceptional generalized geometry which makes the U-duality group manifest and naturally incorporates the scalar degrees of freedom arising in the Ramond-Ramond sector. Via this formalism we derive the structure of the moduli spaces as it is expected from $N = 4$ supergravity.

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Contents

1 Introduction

One class of backgrounds which are commonly considered in string theory are of the form

$$M_{1,d-1} \times Y_{10-d} \, , \tag{1.1}$$

where $M_{1,d-1}$ is a d-dimensional Minkowskian space-time while Y_{10-d} is a compact manifold of dimension $(10-d)$. In such backgrounds the Lorentz group decomposes accordingly into $\mathrm{SO}(1, d-1) \times \mathrm{SO}(10-d)$. The requirement that the background preserves some amount of supersymmetry generically demands that Y_{10-d} admits globally defined and nowhere vanishing spinors. This reduces the structure group G from $\mathrm{SO}(10-d)$ to some subgroup $G \subset \mathrm{SO}(10-d)$.[1] Such manifolds are termed manifolds with G-structure in the literature and they are natural generalizations of Calabi-Yau manifolds [2]–[7].

More precisely, Calabi-Yau manifolds form a subclass of manifolds with G-structure in that the globally defined spinors are also covariantly constant with respect to the Levi-Civita connection. As a consequence the Ricci-tensor vanishes and $G = \mathrm{SU}(n)$ coincides with the holonomy group. However, in the presence of fluxes and localized energy sources such as D-branes and/or orientifold planes the geometry back-reacts and is no longer necessarily Calabi-Yau [8].

In type II theories a slightly more general set-up is possible. The left and the right sector can preserve different "structure groups" and backgrounds with $G \times G$ structure can be considered [9]–[13]. This situation is best described by a formalism — termed generalized geometry — that describes the metric and the B-field of string theory as components of the generalized metric of some bundle E that locally looks like the direct sum of the tangent and the cotangent bundle. On this bundle, the T-duality group $\mathrm{SO}(10-d, 10-d)$ acts naturally, and thus one has an $\mathrm{SO}(10-d, 10-d)$ covariant formulation of string backgrounds. Within this formalism the notion of classical geometry can be relaxed in that Y_{10-d} does not have to be a manifold. Instead objects such as T-folds and generalizations thereof can be considered [14]–[17].[2]

Recently the Ramond-Ramond (RR) sector has also been included in this framework by embedding the T-duality group into the U-duality group $E_{11-d(11-d)}$ [18–21]. The resulting geometrical structure has been termed 'exceptional generalized geometry' as one has to replace the generalized tangent bundle by some exceptional tangent bundle which admits a natural action of $E_{11-d(11-d)}$. The metric of this bundle inherits all scalar degrees of freedom of the low-energy type II theory, including the ones coming from the RR sector.

So far, in type II theories mainly $\mathrm{SU}(3) \times \mathrm{SU}(3)$ backgrounds generalizing Calabi-Yau threefold compactifications have been discussed in the literature. The reason is that they preserve only eight supercharges and therefore are a convenient starting point for particle

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[1]Exceptions of this situation are, for example, in ref. [1]. However, in this paper we do not consider such cases. We thank the referee of our paper for drawing our attention to this possibility.

[2]Throughout this paper we do not specify if Y_{10-d} is an honest manifold or some of the generalizations. For example, we discuss backgrounds where only the tangent space admits a splitting of the form $TM_{1,9} = T_{1,d-1} \oplus F_{10-d}$. Nevertheless we always call Y the compactification manifold and the analysis in this paper just carries over to this more general case.

phenomenology. However, backgrounds with more supersymmetry have also been of considerable interest due to their constraint low energy couplings. In addition, by appropriately orientifolding such backgrounds, one can also construct theories with less supersymmetry.

In this paper we focus on type II backgrounds with 16 supercharges (corresponding to $N = 4$ in $d = 4$) which we discuss in the Minkowskian dimensions $d = 6, 5, 4$. These backgrounds feature the manifolds $Y_{4,5,6}$ which have $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure and which are generalizations of $K3, K3 \times S^1$ and $K3 \times T^2$ respectively. Aspects of such manifolds were previously discussed for example in refs. [7, 12, 22–24], while $N = 4$ compactifications with background fluxes have been analyzed for instance in refs. [25].

One of our main interests in this paper is the relation with $N = 4$ supergravity, which constrains the low energy couplings. For example, the scalar fields of type IIA have to parameterize cosets of the form [26]

$$\mathcal{M} = \frac{\mathrm{SO}(10 - d, n_V)}{\mathrm{SO}(10 - d) \times \mathrm{SO}(n_V)} \times \mathbb{R}^+ \,, \qquad (1.2)$$

where n_V counts the number of vector multiplets and the $\mathbb{R}^+$ factor corresponds to the dilaton.[3] In this paper we identify $\mathcal{M}$ as the deformation space of certain $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure manifolds. For a special class of $\mathrm{SU}(2)$ structure backgrounds in $d = 4$ this was already discussed in [23]. Here we analyze the generic situation and concentrate on the scalar field space which corresponds to the kinetic terms of the scalars in the low energy effective Lagrangian. A more detailed derivation of this Lagrangian including the possible gaugings and the potentials will be presented elsewhere [27]. We choose a possible warp factor in (1.1) to be constant although most of our analysis carries over to the case of a non-trivial warp factor.

In order to analyze manifolds with $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure we use the pure spinor formalism within the framework of generalized geometries [2]. We find that generic $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures do not exist but instead only manifolds with a single $\mathrm{SU}(2)$ structure or with an identity structure can occur. The latter correspond to backgrounds with 32 supercharges, which we do not study any further in this paper. Instead we focus on backgrounds with an honest $\mathrm{SU}(2)$ structure and thus 16 supercharges.

On the class of $\mathrm{SU}(2)$ structure manifolds we impose the additional constraint that at low energies no massive gravitino multiplets survive as their presence would alter the scalar geometry (1.2). Analogously to the analysis of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures in refs. [13] this constraint amounts to projecting out all $\mathrm{SU}(2)$ doublets. In this case we find that the deformations of the $\mathrm{SU}(2)$ structure span the Neveu-Schwarz (NS) subspace of the manifold $\mathcal{M}$ given in (1.2).[4]

The metric on the scalar field space or in other words the kinetic term of the effective action is largely determined by algebraic properties of the pure spinors. In addition differential constraints can be imposed. They appear in the scalar potential and determine

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[3]In $d = 4$ the $\mathbb{R}^+$ factor is enlarged to the coset $Sl(2, \mathbb{R})/\mathrm{SO}(2)$ since the antisymmetric tensor of the NS-sector is dual to an axion and contributes in the scalar couplings.

[4]This NS subspace also appears in corresponding compactifications of the heterotic string on manifolds with $\mathrm{SU}(2)$-structure [28].

possible consistent backgrounds of the theory. However, they also affect the metric in that they select a certain subclass of manifolds with SU(2) structure. For example, we will see that projecting out the doublets can fix a certain component of Y to be related to $K3$.

The RR-sector can be included by using the formalism of exceptional generalized geometry [18–21]. The $SO(10-d, 10-d)$ pure spinors are appropriately embedded into representations of the U-duality group $E_{11-d(11-d)}$, with new degrees of freedom corresponding to the RR-scalars. Due to the RR-sector, type IIA differs from type IIB and we analyze both cases separately. In each case we find agreement with the supergravity results.[5]

The paper is organized as follows. In section 2 we analyze four-dimensional manifolds Y_4 with $SU(2) \times SU(2)$ structure. Before we discuss geometrical and topological issues, we recall in section 2.1 the supergravity field content for type II theories in ten dimensions in an $N = 4$ language. There we see that for $N = 4$ compactifications to be consistent, we have to project out all $SU(2) \times SU(2)$ doublets, which contain the extra gravitino multiplets that are massive in the compactifications. Then we turn to the geometrical description of $SU(2) \times SU(2)$ structures. As a warm-up we start the discussion with (geometric) $SU(2)$ structures in section 2.2. In section 2.3 we then use the pure spinor formalism to describe generic $SU(2) \times SU(2)$ structure backgrounds. With this, we are able to determine the moduli space in section 2.4. In section 2.5 we include the RR-sector by embedding the $SO(4,4)$ pure spinors into an $E_{5(5)} \equiv SO(5,5)$ covariant formulation. This enables us to compute the complete moduli space for type IIA and IIB strings and show the consistency with supergravity.

The more involved case of compactifications to $d = 4$ is discussed in section 3. As in section 2, we start with an analysis of the spectrum in section 3.1, before we turn to a discussion of $SU(2) \times SU(2)$ structures in geometrical terms. After introducing $SU(2)$ structures in section 3.2, we use the pure spinor formalism in section 3.3 to describe the more general case of $SU(2) \times SU(2)$ structures for six-dimensional manifolds Y_6. We find that it is convenient to introduce a generalized almost product structure which reduces the structure group from $SU(3) \times SU(3)$ to $SU(2) \times SU(2)$. We show in section 3.4 that this generalized almost product structure cannot vary and that the moduli space essentially reduces to the one of section 2.4. In section 3.5 we embed the pure spinors in an $E_{7(7)}$ covariant formalism, which we use in section 3.6 to show that the moduli space of type II theories is given by (1.2).

Section 4 contains our conclusions and some of the more technical computations are assembled in 5 appendices. In appendix A we briefly state our conventions and give explicitly the Fierz identities that are used in section 2.2 and 3.2. Appendix B discusses the different cones over the moduli space which appear in the associated superconformal supergravities. Specifically, appendix B.1 is devoted to the hyperkähler cone (or the Swann bundle) [30, 31] which exists over the NS moduli space since it can be viewed as the scalar field space of a theory with eight supercharges. In particular we discuss the relationship of the purity and compatibility conditions with an $SU(2)$ quotient of flat space and express

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[5]Note that there is an intriguing relation between the supergravity moduli spaces and the horizon geometry of extremal black hole attractors [29].

the hyperkähler potential in terms of geometric quantities. In appendix B.2 we discuss the flat cones which arise in the superconformal supergravity of theories with 16 supercharges. In appendix C we give the proof that the generalized almost product structure which we introduced in section 3.3 to define $SU(2) \times SU(2)$ structures on backgrounds of dimension six is rigid for $N = 4$ compactifications. In appendix D we give the details on the calculations in the $E_{7(7)}$ covariant formalism used in section 3.5 and section 3.6. Finally, in appendix E we work out the case of compactifications on $SU(2) \times SU(2)$ structures in $d = 5$, using the strategy of section 2 and section 3. In appendix E.1 we discuss the spectrum of $N = 2$ supergravities in five dimensions achieved by compactification of type II theories on backgrounds of $SU(2) \times SU(2)$ structure. In appendix E.2, we introduce $SU(2)$ structures in five dimensions, and generalize to $SU(2) \times SU(2)$ structures by use of the pure spinor formalism in appendix E.3. There we also derive the NS moduli space. In appendix E.4 we use again the concept of exceptional generalized geometry to include the RR-sector and to compute the complete moduli space. Appendix E has some overlap with the independent work of ref. [20].

2 Compactifications on manifolds with $SU(2) \times SU(2)$ structure in $d = 6$

In this section we study backgrounds of the form (1.1) with a six-dimensional Minkowskian space-time $M_{1,5}$ times a four-dimensional manifold Y_4. We focus on the situation where these backgrounds preserve $N = 2$ in $d = 6$. Since the spinor representation is eight-dimensional in $d = 6$ this corresponds to 16 supercharges or $N = 4$ supersymmetry in terms of a four-dimensional counting.

More precisely the Lorentz group for these backgrounds is $SO(1,5) \times SO(4)$. The ten-dimensional spinor representation decomposes accordingly as

$$\mathbf{16} \to (\mathbf{4}, \mathbf{2}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{2}}), \tag{2.1}$$

where the $\mathbf{4}$ is a Weyl spinor of $SO(1,5)$ while the $\mathbf{2}$ denotes the spinor representation of $SO(4)$.

Preserving half of the supercharges amounts to choosing backgrounds that admit one or two globally defined spinors, which corresponds to manifolds Y_4 with a reduced structure group $SU(2)$ or $SU(2) \times SU(2)$, respectively.[6] Let us now discuss both cases in turn. We will start with a general analysis of the spectrum of type II supergravities in such backgrounds, and then introduce a proper geometrical description that allows us to determine the structure of the moduli space explicitly.

2.1 Field decompositions and spectrum

In this section we give the massless type II supergravity fields in terms of their representations under the six-dimensional Lorentz group and the structure group and analyze how they assemble in $N = 2$ multiplets, in the spirit of [13].

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[6]In the following, whenever we refer to an object as being globally defined we mean globally defined and nowhere vanishing.

Because we deal with massless particles, we can use the light-cone gauge where we only have to consider representations of $SO(4)_{lc}$ instead of the whole $SO(1,5)$ Lorentz group. Since we treat the case of an $SU(2) \times SU(2)$ structure group, we examine the decomposition of massless type II fields under the group $SO(4)_{lc} \times SU(2) \times SU(2)$. For this, let us recall the decomposition of the two Majorana-Weyl representations $\mathbf{8}^s$ and $\mathbf{8}^c$ and the vector representation $\mathbf{8}^v$ under the breaking

$$SO(8) \to SO(4)_{lc} \times SO(4) \to SO(4)_{lc} \times SU(2) \,. \tag{2.2}$$

We get

$$\begin{aligned}
\mathbf{8}^s &\to \mathbf{2_2} \oplus \mathbf{\bar{2}_{\bar{2}}} \to \mathbf{2_2} \oplus 2\,\mathbf{1_{\bar{2}}} \,, \\
\mathbf{8}^c &\to \mathbf{2_{\bar{2}}} \oplus \mathbf{\bar{2}_2} \to \mathbf{2_{\bar{2}}} \oplus 2\,\mathbf{1_2} \,, \\
\mathbf{8}^v &\to \mathbf{1_4} \oplus \mathbf{4_1} \to \mathbf{1_4} \oplus 2\,\mathbf{2_1} \,,
\end{aligned} \tag{2.3}$$

where the subscript denotes the representation under the group $SO(4)_{lc}$.

In type IIA string theory the massless fermionic degrees of freedom originate from the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^c)$ representation of $SO(8)_L \times SO(8)_R$, while in type IIB they originate from the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^s)$ representation. Respectively, under the decomposition $SO(8)_L \times SO(8)_R \to SO(4)_{lc} \times SU(2)_L \times SU(2)_R$ we find

$$\begin{aligned}
(\mathbf{8}^s, \mathbf{8}^v) &\to 2(\mathbf{1},\mathbf{1})_\mathbf{6} \oplus 2(\mathbf{1},\mathbf{1})_\mathbf{2} \oplus (\mathbf{1},\mathbf{2})_\mathbf{\bar{6}} \oplus (\mathbf{1},\mathbf{2})_\mathbf{\bar{2}} \oplus 4(\mathbf{2},\mathbf{1})_\mathbf{\bar{2}} \oplus 2(\mathbf{2},\mathbf{2})_\mathbf{2} \,, \\
(\mathbf{8}^v, \mathbf{8}^s) &\to 2(\mathbf{1},\mathbf{1})_\mathbf{6} \oplus 2(\mathbf{1},\mathbf{1})_\mathbf{2} \oplus 4(\mathbf{1},\mathbf{2})_\mathbf{\bar{2}} \oplus (\mathbf{2},\mathbf{1})_\mathbf{\bar{6}} \oplus (\mathbf{2},\mathbf{1})_\mathbf{\bar{2}} \oplus 2(\mathbf{2},\mathbf{2})_\mathbf{2} \,, \\
(\mathbf{8}^v, \mathbf{8}^c) &\to 2(\mathbf{1},\mathbf{1})_\mathbf{\bar{6}} \oplus 2(\mathbf{1},\mathbf{1})_\mathbf{\bar{2}} \oplus 4(\mathbf{1},\mathbf{2})_\mathbf{2} \oplus (\mathbf{2},\mathbf{1})_\mathbf{6} \oplus (\mathbf{2},\mathbf{1})_\mathbf{2} \oplus 2(\mathbf{2},\mathbf{2})_\mathbf{\bar{2}} \,.
\end{aligned} \tag{2.4}$$

We see that half of the gravitinos, denoted by the subscript $\mathbf{6}$ and $\mathbf{\bar{6}}$, come in the $(\mathbf{1},\mathbf{1})$ representation while the other half is in the doublet representations $(\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1})$ of $SU(2)_L \times SU(2)_R$. We will see below that the 6d graviton is in the $(\mathbf{1},\mathbf{1})$ representation and thus this representation refers to the gravity multiplet in six dimensions. Hence the $(\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1})$ representations correspond to additional gravitino multiplets, which acquire a mass at the Kaluza-Klein scale due to the $SU(2) \times SU(2)$ structure background. We have to project out these representations to end up with a standard $N = 2$ supergravity in six dimensions [32]. This is analogous to projecting out all $SU(3) \times SU(3)$ triplets in [13] to achieve a standard $N = 2$ theory in four dimensions. After this projection, the fermionic components in the $(\mathbf{1},\mathbf{1})$ representation become part of the gravity multiplet, while the $(\mathbf{2},\mathbf{2})$ components correspond to the fermionic degrees of freedom in the $N = 2$ vector and tensor multiplets in type IIA and IIB, respectively.[7]

The massless bosonic fields of type II supergravity can be decomposed analogously. For the NS-NS-sector, it is convenient to consider the combination

$$E_{MN} = g_{MN} + B_{MN} + \phi\, \eta_{MN} \,, \tag{2.5}$$

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[7]In type IIB, only the anti-self-dual part of the antisymmetric two-tensor is part of the gravity multiplet [33]. The self-dual component forms a tensor multiplet together with scalars in the RR-sector. This tensor multiplet is also in the $(\mathbf{1},\mathbf{1})$ representation.

which decomposes as

$$
\begin{aligned}
E_{\mu\nu} &: (\mathbf{1},\mathbf{1})_{\mathbf{9}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{1}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{3}\oplus\bar{\mathbf{3}}} \,, \\
E_{\mu m} &: 2(\mathbf{1},\mathbf{2})_{\mathbf{4}} \,, \\
E_{m\mu} &: 2(\mathbf{2},\mathbf{1})_{\mathbf{4}} \,, \\
E_{mn} &: 4(\mathbf{2},\mathbf{2})_{\mathbf{1}} \,.
\end{aligned}
\tag{2.6}
$$

Projecting out the doublets eliminates the six-dimensional vectors $E_{\mu m}$ and $E_{m\mu}$, and we are left with $E_{\mu\nu}$, i.e. the metric, the six-dimensional dilaton and the antisymmetric two-tensor, which are part of the gravity multiplet, and the scalars E_{mn}, which reside in vector or tensor multiplets. Since the latter ones correspond to the internal metric and B-field components, they can be associated with deformations of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ background.

Finally, in the RR-sector we need to decompose the $(\mathbf{8}^s, \mathbf{8}^c)$ representation in type IIA and the $(\mathbf{8}^s, \mathbf{8}^s)$ in type IIB. One finds

$$
\begin{aligned}
(\mathbf{8}^s, \mathbf{8}^c) &\to 4(\mathbf{1},\mathbf{1})_{\mathbf{4}} \oplus 2(\mathbf{1},\mathbf{2})_{\bar{\mathbf{3}}} \oplus 2(\mathbf{1},\mathbf{2})_{\mathbf{1}} \oplus 2(\mathbf{2},\mathbf{1})_{\mathbf{3}} \oplus 2(\mathbf{2},\mathbf{1})_{\mathbf{1}} \oplus (\mathbf{2},\mathbf{2})_{\mathbf{4}} \,, \\
(\mathbf{8}^s, \mathbf{8}^s) &\to 4(\mathbf{1},\mathbf{1})_{\bar{\mathbf{3}}} \oplus 4(\mathbf{1},\mathbf{1})_{\mathbf{1}} \oplus 2(\mathbf{1},\mathbf{2})_{\mathbf{4}} \oplus 2(\mathbf{2},\mathbf{1})_{\mathbf{4}} \oplus (\mathbf{2},\mathbf{2})_{\mathbf{3}} \oplus (\mathbf{2},\mathbf{2})_{\mathbf{1}} \,.
\end{aligned}
\tag{2.7}
$$

We see that in type IIA only six-dimensional vectors in the RR-sector survive the projection. Those which are in the $(\mathbf{1},\mathbf{1})$ representation form the graviphotons in the gravity multiplet, those in the $(\mathbf{2},\mathbf{2})$ give the vectors in the vector multiplets.

Projecting out all $\mathrm{SU}(2) \times \mathrm{SU}(2)$ doublets leaves a spectrum that for type IIA combines into a gravitational multiplets plus a vector multiplet of the non-chiral $d = 6, N = 2$ supergravity. For type IIB we obtain instead a gravitational multiplets and two tensor multiplet of the chiral $N = 2$ supergravity. To be more precise, in type IIA the gravitational multiplet contains the graviton, an antisymmetric tensor, two (non-chiral) gravitini, four vector fields, four Weyl fermions and a real scalar. These degrees of freedom precisely correspond to the $(\mathbf{1},\mathbf{1})$ representation of the decompositions given in (2.4), (2.6) and (2.7). The vector multiplet contains a vector field, four gaugini and four real scalars. These arise in the $(\mathbf{2},\mathbf{2})$ representation of the above decompositions. In type IIB the gravitational multiplet contains the graviton, five self-dual antisymmetric tensor and two (chiral) gravitini. These degree of freedom are found in the $(\mathbf{1},\mathbf{1})$ representation of the above decompositions. In addition there are two tensor multiplets each containing an anti-self-dual antisymmetric tensor, two chiral fermions and five scalars. One of them also originates from the $(\mathbf{1},\mathbf{1})$ representation while the second one comes from the $(\mathbf{2},\mathbf{2})$ representation of the above decompositions.

Note that the resulting fields still depend on all coordinates of the ten-dimensional spacetime, i.e. we have not performed any Kaluza-Klein truncation on the spectrum but really deal with ten-dimensional backgrounds. This procedure just corresponds to a rewriting of the ten-dimensional supergravity in a form where instead of the ten-dimensional Lorentz group only $\mathrm{SO}(1,5) \times \mathrm{SO}(4)$ with sixteen supercharges is manifest. This "rewriting" of the ten-dimensional theory has been pioneered in ref. [34] and applied to the case of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures in refs. [13, 35].

2.2 General remarks on $SU(2)$ structures

In this section we review some general facts about the geometry of $SU(2)$ structures on four-dimensional manifolds Y_4. A prominent example of such manifolds is $K3$ which even has $SU(2)$ holonomy (for review, see e.g. [36]).

The requirement of unbroken supercharges demands that the internal manifold admits a globally defined, nowhere vanishing spinor η. Four-dimensional manifolds with this property have a structure group G that is contained in $SU(2)$ and are called manifolds of $SU(2)$ structure. This is due to the fact that η defines an element of the $SO(4)$ spinor bundle which is a singlet of the structure group $G \subset SO(4)$. Since the spin bundle of $SO(4)$ is $SU(2) \times SU(2)$, the choice of the spinor η is equivalent to the choice of a particular unbroken $SU(2)$ subgroup of $SO(4)$, which is identified with the structure group $G = SU(2)$.

Let us now consider the decomposition of the spinor representations under G. The defining spinor η and its charge conjugate η^c are both globally defined and nowhere vanishing and therefore they are both singlets under the structure group G. Moreover, they are linearly independent and have the same chirality.[8] Together they span the space of Weyl spinors of given chirality, which thus is decomposed into two $SU(2)$ singlets. The Weyl spinor representation of opposite chirality forms a doublet under the $SU(2)$ structure group. It is just a matter of convention which representation we denote by $\mathbf{2}$ and which one by $\bar{\mathbf{2}}$. Here and in the following we choose

$$\mathbf{2} \to \mathbf{2}, \qquad \bar{\mathbf{2}} \to \mathbf{1} \oplus \mathbf{1} \qquad (2.8)$$

for the breaking $SO(4) \to SU(2)$.

From the two singlets one can construct three distinct globally defined two-forms by appropriately contracting with $SO(4)$ γ-matrices [7]

$$\bar{\eta} \gamma_{mn} \eta = -\mathrm{i}\, J_{mn}, \quad \bar{\eta}^c \gamma_{mn} \eta = \mathrm{i}\, \Omega_{mn}, \quad \bar{\eta} \gamma_{mn} \eta^c = \mathrm{i}\, \bar{\Omega}_{mn}, \quad m, n = 1, \dots, 4, \qquad (2.9)$$

where the normalization $\bar{\eta}\eta = 1$ is chosen. However, these two-forms are not independent but satisfy

$$\Omega \wedge \bar{\Omega} = 2J \wedge J \neq 0, \qquad \Omega \wedge J = 0, \qquad \Omega \wedge \Omega = 0, \qquad (2.10)$$

which follows from the Fierz identities given in (A.3). Conversely, the Fierz identities also show that the choice of a real two-form J and a complex two-form Ω determines η completely (up to normalization) if they satisfy the above relations. Therefore, J and Ω equivalently define an $SU(2)$ structure on the manifold.

Alternatively one can also define an $SU(2)$-structure in terms of stable forms [3]. A stable p-form $\omega \in \Lambda^p V^*$ on a vector space V is defined as a form whose orbit under the action of $Gl(V)$ is open in $\Lambda^p V^*$. For a stable two-form ω on a $2m$-dimensional space this means that $\omega^m \neq 0$. Thus, a stable two-form on an even-dimensional space defines a symplectic structure on it.

On a four-dimensional manifold Y_4 the stable two-form J satisfies $J \wedge J \sim \mathrm{vol}_4$ and locally defines a symplectic structure that reduces the structure group from $Gl(4)$

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[8]Note that η and η^c have the same chirality in four Euclidean dimensions (see appendix A). The transposed spinors, i.e. η^t and $\bar{\eta} = (\eta^c)^t$ are also globally defined and are the singlets of the dual space.

to $Sp(4, \mathbb{R})$.[9] The existence of additional stable forms can reduce the structure group even further. In this case one has to ensure that these stable forms do not reduce the structure group in the same way. For example, one can take two linearly independent stable two-forms $J_i, i = 1, 2$ that satisfy

$$J_i \wedge J_j = \delta_{ij} \, \text{vol}_4 \ . \tag{2.11}$$

J_1 and J_2 then define a holomorphic two-form $\Omega = J_1 + \mathrm{i}\, J_2$, which globally defines a holomorphic subbundle in the tangent space and therefore breaks the structure group to $Sl(2, \mathbb{C}) \equiv Sp(2, \mathbb{C})$.

Analogously, in the case of three stable two-forms $J_i, i = 1, 2, 3$, which satisfy (2.11) the structure group is reduced even further. Since J_3 is orthogonal to $\Omega = J_1 + \mathrm{i}\, J_2$ and its complex conjugate, it defines a product between the holomorphic and the anti-holomorphic tangent bundle. Therefore, the $Sl(2, \mathbb{C})$ is further broken to the SU(2) subgroup which preserves this product.[10] If one defines

$$J = J_3 , \quad \Omega = J_1 + \mathrm{i}\, J_2 , \tag{2.12}$$

it is straightforward to check that (2.11) and (2.10) are indeed equivalent.

In terms of stable forms it is easy to identify the parameter space of SU(2) structures. A triple of stable forms J_i has to fulfill (2.11) in order to define an SU(2) structure on Y_4. Thus the J_i span a three-dimensional subspace in the space of two-forms. By choosing some volume form vol_4, i.e. some orientation on Y_4, we can interpret the wedge product as a scalar product of split signature on the space of two-forms. With respect to this scalar product, the J_i form an orthonormal basis for a space-like subspace. The SO(3, 3) orbit of such a triple of J_i gives all possible configurations that respect (2.11). Thus, the configuration space can be written as SO(3, 3) divided by the stabilizer of the J_i. The stabilizer consists of SO(3) rotations in the subspace orthogonal to the J_i, which leave the SU(2) structure invariant. Therefore, the configuration space for the J_i is SO(3, 3)/SO(3). The SO(3) rotations in the stabilizer correspond to the action of the SU(2) structure group on the space of two-forms. The J_i are singlets under the SU(2) structure group while the space orthogonal to them forms an SU(2) triplet.

One should note that there is some redundancy in the descriptions of SU(2) structures on a manifold. Any rescaling of the J_i does not change the unbroken SU(2) and therefore does not correspond to a degree of freedom for the SU(2) structure. Hence we can fix the normalization by (2.11). Furthermore, there is a rotational SO(3) symmetry between the three forms J_i. However, this symmetry is not obvious from the definition (2.9). It corresponds to SU(2) rotations on the Weyl-spinor doublet (η, η^c) which is a symmetry because η and η^c have the same chirality on a four-dimensional manifold. One can check that the three two-forms J_i indeed form a triplet under the action of this SU(2). By modding out this symmetry, we arrive at the parameter space of an SU(2) structure over

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[9]Note that this symplectic structure may be non-integrable in the sense that $\mathrm{d}\omega \neq 0$. Therefore, our notion of a symplectic structure differs from the usual mathematical terminology.

[10]Of course, this breaking is just the well-known relation $Sl(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}) = \text{SU}(n)$.

a point on the manifold Y_4, which is

$$\mathcal{M}_{J_i} = \frac{\mathrm{SO}(3,3)}{\mathrm{SO}(3) \times \mathrm{SO}(3)} \ . \tag{2.13}$$

By consideration of the corresponding spin groups of $\mathrm{SO}(3,3)$ and $\mathrm{SO}(3)$ one expresses this result as

$$\mathcal{M}_{J_i} = \frac{Sl(4,\mathbb{R})}{\mathrm{SO}(4)} \ . \tag{2.14}$$

If one compares this with the parameter space $Gl(4,\mathbb{R})/\mathrm{SO}(4)$ of the metric over a point of Y_4, we see that the parameter space of $\mathrm{SU}(2)$ structures incorporates all metric degrees of freedom except the volume factor. The missing degree of freedom corresponding to the volume factor can be associated with the normalization of the J_i in (2.11).[11]

At the end of section 2.4 we derive the global moduli space of an $\mathrm{SU}(2)$ structure over Y_4 from (2.13). However, before we do so, we first generalize our local discussion to general $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures.

2.3 $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures and pure spinors

One can generalize the $\mathrm{SU}(2)$ structures discussed in the previous section by assuming that the manifold admits two globally defined, nowhere vanishing spinors η_1 and η_2. Each of them defines an $\mathrm{SU}(2)$ structure on its own and if they are identical everywhere on the manifold, this reduces to the case discussed in the previous section. In the other limiting case where η_1 and η_2 are orthogonal at each point, the two $\mathrm{SU}(2)$ structures intersect in some identity structure, which means that the spinor bundle is trivial and compactification on this backgrounds preserves all 32 supercharges (corresponding to $N = 8$ in $d = 4$). However, in principle one can also have the intermediate case of two globally defined, nowhere vanishing spinors η_1 and η_2 that are linearly independent at most points but become parallel at some points on the manifold.

Analogously to the last section, there is an equivalent formulation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures in terms of globally defined stable forms. This is elegantly captured by the notion of pure spinors and generalized geometry [6, 9, 37]. Let us briefly review this concept for a $2n$-dimensional manifold Y and then return to the case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures afterwards.

2.3.1 Generalized geometry and the pure spinor approach

In generalized geometry one considers a generalized tangent bundle $\mathcal{T}Y$ which locally looks like $TY \oplus T^*Y$ and therefore admits a scalar product $\mathcal{I}$ of split signature that is induced by the canonical pairing between tangent and cotangent space. On a $2n$-dimensional manifold Y this bundle thus has a structure group contained in $\mathrm{SO}(2n, 2n)$.

---

[11]Note that the choice of a triple of normalized J_i is just equivalent to the choice of a Hodge operator $*$ on the space of two-forms. This is reflected by the fact that the J_i just span the positive eigenspace of $*$ in the space of two-forms.

Similar to our discussion in the last section, one can introduce further objects that break the structure group. For example, an almost complex structure $\mathcal{J}$, fulfilling

$$\mathcal{J}^2 = -1 \,, \tag{2.15}$$

can be defined if (and only if) the complexified generalized tangent bundle globally splits as

$$(\mathcal{T}Y)_{\mathbb{C}} = L_+ \oplus L_- \,, \tag{2.16}$$

where $L_{\pm}$ are the eigenspaces of $\mathcal{J}$ with the eigenvalues $\pm \mathrm{i}$. If $\mathcal{J}$ is globally defined on Y, the structure group of $\mathcal{T}Y$ is broken from $\mathrm{SO}(2n, 2n)$ to $\mathrm{U(n, n)}$.

When two generalized almost complex structures $\mathcal{J}_{1,2}$ exist, the notion of compatibility can be defined. More precisely, $\mathcal{J}_1$ and $\mathcal{J}_2$ are called compatible if

1. $\mathcal{J}_1$ and $\mathcal{J}_2$ commute and

2. $\mathcal{G} := \mathcal{I}\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $\mathcal{T}Y$, where $\mathcal{I}$ is the canonical scalar product on $\mathcal{T}Y$.

The first condition ensures that the splittings (2.16) can be done simultaneously, i.e. that

$$(\mathcal{T}Y)_{\mathbb{C}} = L_{++} \oplus L_{-+} \oplus L_{+-} \oplus L_{--} \,, \tag{2.17}$$

where the indices correspond to the eigenvalues of $\mathcal{J}_{1,2}$. The second condition ensures that each of the four components in (2.17) is n-dimensional such that the two compatible generalized almost complex structures reduce the structure group to $\mathrm{U(n)} \times \mathrm{U(n)}$, where each $\mathrm{U(n)}$ acts on two of the four components.

Let us now briefly review how to reformulate generalized geometry in terms of pure spinors Φ [6, 9]. One first defines the annihilator space L_{Φ} of a complex $\mathrm{SO}(2n, 2n)$ Weyl spinor Φ as the subspace of complexified gamma-matrices which map Φ to zero, i.e.

$$L_{\Phi} \equiv \{\Gamma \in (\mathcal{T}Y)_{\mathbb{C}} \,|\, \Gamma\Phi = 0\} \,. \tag{2.18}$$

Note that L_{Φ} is always isotropic as for each element Γ of L_{Φ} we have

$$0 = \Gamma^2\Phi = \mathcal{I}(\Gamma, \Gamma)\,\Phi \,, \tag{2.19}$$

which implies $\mathcal{I}(\Gamma, \Gamma) = 0$ for all $\Gamma \in L_{\Phi}$.

A complex Weyl spinor Φ of $\mathrm{SO}(2n, 2n)$ is called pure if its annihilator space has maximal dimension, i.e. $\dim L_{\Phi} = 2n$. Φ is called normalizable if

$$\langle \Phi, \bar{\Phi} \rangle > 0 \,, \tag{2.20}$$

where the brackets denote the usual spinor product. As a consequence of Chevalley's theorem [38], which states

$$\dim L_{\Phi} \cap L_{\Psi} = 0 \quad \Leftrightarrow \quad \langle \Phi, \Psi \rangle \neq 0 \,, \tag{2.21}$$

normalizable pure spinors define a splitting

$$(\mathcal{T}Y)_{\mathbb{C}} = L_\Phi \oplus L_{\bar{\Phi}} \ . \tag{2.22}$$

By matching the annihilator space L_Φ with the $+\mathrm{i}$ eigenspace of some generalized almost complex structure, one can show that both are equivalent up to the normalization factor of Φ. Thus, generalized almost complex structures are equivalent to lines of pure $\mathrm{SO}(2n, 2n)$ spinors. A pure spinor therefore breaks the structure group of $\mathcal{T}Y$ further from $\mathrm{U}(n, n)$ to $\mathrm{SU}(n, n)$ in that fixing its phase eliminates the $\mathrm{U}(1)$ factor.

The compatibility conditions for two generalized almost complex structures translate into a compatibility condition on the corresponding pure spinors. Two normalizable pure $\mathrm{SO}(2n, 2n)$ spinors $\Phi_{1,2}$ are compatible if and only if their annihilator spaces intersect in a space of dimension n, i.e.

$$\dim(L_{\Phi_1} \cap L_{\Phi_2}) = n \ . \tag{2.23}$$

Thus the pair $\Phi_{1,2}$ breaks the structure group to $\mathrm{SU}(n) \times \mathrm{SU}(n)$ (instead of $\mathrm{U}(n) \times \mathrm{U}(n)$). Therefore, pure spinors of generalized geometry provide a convenient framework to deal with $\mathrm{SU}(n) \times \mathrm{SU}(n)$ structures. Whenever $\mathcal{T}Y = TY \oplus T^*Y$ globally, both $\mathrm{SU}(n)$ factors can be projected to the tangent space TY. In this case the intersection of these projections defines the structure group of the tangent bundle.

The compatibility condition of two pure spinors also restricts their chirality. Since $\mathrm{SO}(n, n)$ transformations do not mix chiralities, one can always assume Φ_1 and Φ_2 to be of definite chirality. Furthermore, two pure spinors Φ_1 and Φ_2 have the same chirality if and only if [39]

$$\dim(L_{\Phi_1} \cap L_{\Phi_2}) = 2k \tag{2.24}$$

for $k \in \mathbb{N}$. Therefore, two compatible pure spinors are of the same chirality if n is even and of opposite chirality for n being odd.

One can construct pure $\mathrm{SO}(2n, 2n)$ spinors out of the two globally defined $\mathrm{SO}(2n)$ spinors η_1 and η_2 discussed at the beginning of this section. More precisely, one has

$$\eta_1 \otimes \bar{\eta}_2 = \frac{1}{4} \sum_{k=0}^{2n} \frac{1}{k!} \left(\bar{\eta}_2 \gamma_{m_1 \dots m_k} \eta_1 \right) \gamma^{m_k \dots m_1} \ , \tag{2.25}$$

where $\gamma^{m_1 \dots m_k}$ is the totally antisymmetric product of $\mathrm{SO}(2n)$ γ-matrices. One can act with $\mathrm{SO}(2n)$ gamma-matrices from the left or from the right which in turn defines an $\mathrm{SO}(2n, 2n)$ action on the bi-spinor $\eta_1 \otimes \bar{\eta}_2$. By extensive use of the Fierz identities given in appendix A, one can show that $\eta_1 \otimes \bar{\eta}_2$ is pure and normalizable. The same holds for $\eta_1 \otimes \bar{\eta}_2^c$ and these two pure spinors are moreover compatible. Thus they can be used to discuss $\mathrm{SU}(n) \times \mathrm{SU}(n)$ structures.

The map

$$\tau \ \eta_1 \otimes \bar{\eta}_2 \longmapsto \tau(\eta_1 \otimes \bar{\eta}_2) \equiv \frac{1}{4} \sum_{k=0}^{2n} \frac{1}{k!} \left(\bar{\eta}_2 \gamma_{m_1 \dots m_k} \eta_1 \right) e^{m_k} \wedge \dots \wedge e^{m_1} \ , \tag{2.26}$$

with e^{m_k} being a local basis of one-forms, identifies $\mathrm{SO}(2n, 2n)$ spinors with formal sums of differential forms. This isomorphism is canonical up to the choice of a volume form on

the manifold [13]. Note that (2.26) maps negative (positive) chirality spinors to differential forms of odd (even) degree. Moreover, it is an isometry with respect to the spinor product and the so-called Mukai pairing. The latter is defined by[12]

$$\langle \Psi, \chi \rangle = \sum_p (-)^{[(p-1)/2]} \Psi_p \wedge \chi_{2n-p}, \tag{2.27}$$

which maps two formal sums of differential forms to a form of top degree. Like the spinor product, it is symmetric for n even, and anti-symmetric for n being odd. Using the definition

$$\lambda \alpha_p = (-1)^{[(p-1)/2]} \alpha_p \tag{2.28}$$

for a p-form α_p, we can write the Mukai pairing also in the form

$$\langle \Psi, \chi \rangle = [\Psi \wedge \lambda \chi]_{\deg=2n} . \tag{2.29}$$

In the following we frequently use the isomorphism (2.26).

Before we go on, let us state two important facts. One can show that a pure spinor Φ is always of the form [9]

$$\Phi = e^{-B} \wedge e^{-iJ} \wedge \Omega, \tag{2.30}$$

where B and J are real two-forms and Ω is some complex k-form, $k \leq 2n$, that is locally decomposable into complex one-forms. Furthermore, one can prove that two compatible pure spinors are always of the form [37]

$$\Phi_1 = e^{-B} \wedge \tau(\eta_1 \otimes \bar\eta_2), \qquad \Phi_2 = e^{-B} \wedge \tau(\eta_1 \otimes \bar\eta_2^c), \tag{2.31}$$

where the isomorphism τ is defined in (2.26).

Before we close this section, let us define the generalized Hodge operator [11, 40]

$$*_B = e^{-B} * \lambda\, e^B \tag{2.32}$$

which acts on the space of forms, with λ defined by (2.28). Under the isomorphism given in (2.26) the generalized Hodge operator is mapped to charge conjugation on the space of $SO(2n, 2n)$ spinors. Analogously to the conventional Hodge operator, the generalized version can define a positive definite metric $G(\cdot, \cdot) \equiv \langle \cdot, *_B \cdot \rangle$ on the space of forms, which is just the composition of $*_B$ with the Mukai pairing. From (2.29) it is easy to see that G acts on the space of forms by

$$G(e^{-B} \wedge \Psi, e^{-B} \wedge \chi) = \langle e^{-B} \wedge \Psi, *_B\, e^{-B} \wedge \chi \rangle = [\Psi \wedge *\chi]_{\deg=2n} = \sum_{p=0}^{2n} \Psi_p \wedge *\chi_p, \tag{2.33}$$

which indeed is positive definite. Therefore, the Mukai pairing and the generalized Hodge operator have the same signature.

Since $*_B^2 = 1$ on forms of even degree, we see that the generalized Hodge operator corresponds to an almost product structure on $\Lambda^{\text{even}} T^* Y$. For $B = 0$, it coincides on two-forms with the conventional Hodge star operator, which is of split signature over each point.

---

[12]Here, $[\cdot]$ is the floor function which rounds down its argument to the next integer number.

In this case, the forms $1\pm\text{vol}$ are eigenvectors of $*_B$ to the eigenvalue ∓ 1 and we see that $*_B$ has split signature on $\Lambda^{\text{even}}T^*Y$ over each point of Y. Since B can be continously switched on, the signature is independent of B, and the eigenspaces of $*_B$ (with eigenvalue ± 1) at a given point on the manifold have the same dimension. As the standard Hodge operator, the generalized Hodge operator $*_B$ can be globally defined on Y and therefore must be invariant under the $\text{SU}(2) \times \text{SU}(2)$ structure group. Hence, $*_B$ leaves the $\text{SU}(2) \times \text{SU}(2)$ representations invariant, and its eigenspaces coincide with these representations.

As stated at the beginning of this section the generalized tangent bundle $\mathcal{T}Y$ locally has the structure $TY \oplus T^*Y$. In the following, we first perform a local analysis and consider the algebraic structure of the bundle over a point on the manifold. Therefore, we use the notation $TY \oplus T^*Y$ instead of $\mathcal{T}Y$.

2.3.2 Pure spinors on a manifold of dimension four

Let us now apply the previous discussion to the case of $\text{SU}(2) \times \text{SU}(2)$ structures on a four-dimensional manifold Y_4. However, let us start with the simpler case of $\text{SU}(2)$ structures or in other words with the case where a single $\text{SO}(4)$ spinor η exists on Y_4. With its help we can construct the two pure $\text{SO}(4,4)$ spinors $\eta \otimes \bar\eta$ and $\eta \otimes \bar\eta^c$. Using the definitions (2.9) and (2.26) we identify[13]

$$\tau(\eta \otimes \bar\eta) = \frac{1}{4}\,e^{-\,i\,J}\,, \qquad \tau(\eta \otimes \bar\eta^c) = \frac{1}{4}\,i\,\Omega\,. \tag{2.34}$$

We can additionally shift these pure $\text{SO}(4,4)$ spinors by a B-field leaving all conditions unchanged. Thus, we arrive at

$$\Phi_1 = \frac{1}{4}\,e^{-B-i\,J}\,, \qquad \Phi_2 = \frac{i}{4}\,e^{-B} \wedge \Omega\,. \tag{2.35}$$

Let us now turn to the case of general $\text{SU}(2) \times \text{SU}(2)$ structures. We first analyze the conditions for spinors to be pure and compatible. For the case at hand this is simplified by the triality property of $\text{SO}(4,4)$ which isomorphically permutes the spinor representation, its conjugate and the vector representation among each other. In particular, the quadratic form $\langle \cdot, \cdot \rangle$ on the spinor space is mapped to the usual scalar product on the vector space. This fact is used in the following.

For a pure spinor Φ the annihilator space L_Φ has dimension four. In addition Chevalley's theorem (2.21) implies

$$\langle \Phi, \Phi \rangle = 0\,. \tag{2.36}$$

As shown in [39], this condition is also sufficient for Φ to be pure. Thus, pure $\text{SO}(4,4)$ spinors correspond to complex light-like vectors under the triality map. Furthermore, for Φ to be normalizable we need

$$\langle \Phi, \bar\Phi \rangle > 0\,. \tag{2.37}$$

---

[13]This can be seen from the definition (2.9) together with the normalization $\bar\eta\eta = 1$ and the fact that η and η^c are orthogonal, i.e. $\eta^t\eta = 0$. Furthermore, using the Fierz identities (A.3) one can show $J \wedge J = J \wedge *J = \text{vol}_4$ which allows us to write the first tensor product as an exponential.

Now let us consider two pure normalizable spinors Φ_i, $i = 1, 2$ which by definition satisfy

$$\langle \Phi_i, \Phi_i \rangle = 0 , \qquad \langle \Phi_i, \bar{\Phi}_i \rangle > 0 . \tag{2.38}$$

If they are compatible, they also satisfy (2.23), which on Y_4 reads

$$\dim(L_{\Phi_1} \cap L_{\Phi_2}) = 2 . \tag{2.39}$$

From (2.24) we conclude that both Φ_i have the same chirality which, together with (2.21), implies that (2.39) is equivalent to

$$\langle \Phi_1, \Phi_2 \rangle = 0 , \qquad \langle \Phi_1, \bar{\Phi}_2 \rangle = 0 . \tag{2.40}$$

Finally, we can choose the normalization

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle \neq 0 . \tag{2.41}$$

Let us close this section by analyzing which possible cases of $SU(2) \times SU(2)$ structures can occur on four-dimensional manifolds. We just argued that Φ_1 and Φ_2 have the same chirality so that the corresponding forms are of odd or even degree. We start with the case where both spinors Φ_1 and Φ_2 have negative chirality. From (2.30) we see that both pure spinors are of the form

$$\Phi_i = U_i \wedge e^{-i J_i} , \quad i = 1, 2 , \tag{2.42}$$

where U_i are two complex one-forms while J_i are two non-vanishing real two-forms.[14] In addition, the compatibility condition (2.40) implies

$$U_1 \wedge U_2 \wedge (J_1 - J_2) = 0 , \qquad U_1 \wedge \bar{U}_2 \wedge (J_1 + J_2) = 0 , \tag{2.43}$$

while the normalization (2.41) translates into

$$U_1 \wedge \bar{U}_1 \wedge J_1 = U_2 \wedge \bar{U}_2 \wedge J_2 \neq 0 . \tag{2.44}$$

Since $U_2 = aU_1 + b\bar{U}_1$ does not solve (2.43) and (2.44) we conclude that U_2 is linearly independent of U_1 and $\bar{U}_1$ and therefore $U_1, \bar{U}_1, U_2, \bar{U}_2$ form a basis of T^*Y_4. Thus, we can find four one-forms that form a basis at every point of Y_4 and hence the manifold Y is parallelizable. This means that the two factors of the $SU(2) \times SU(2)$ structure just intersect in the identity. Thus, the structure group of the manifold is trivial. This in turn implies that Y_4 admits four globally defined $SO(4)$ spinors corresponding to string backgrounds with 32 supercharges. This fact can also be seen from (2.31). Since $\Phi_{1,2}$ are of odd degree, η_1 and η_2 are of opposite chirality. Together with their charge conjugated spinors they lead to four globally defined spinors. Since in this paper we focus on backgrounds with 16 supercharges, we do not discuss this case any further.

Let us turn to the case where both spinors are of even degree. The most general form for those two spinors is given in (2.31), where now η_1 and η_2 are of the same chirality to ensure that Φ_1 and Φ_2 are of even degree. As we explained above eq. (2.8) a spinor η_1 and

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[14]For simplicity we ignore the B-field which however can be easily included.

its charge conjugate η_1^c span the whole space of Weyl spinors of a given chirality. Therefore, η_2 has to be a linear combination of η_1 and η_1^c. However, this means that we can rotate Φ_1 and Φ_2 in such a way that they are of the form

$$\Phi_1 = \mathrm{e}^{-B} \wedge \tau(\eta_1 \otimes \bar{\eta}_1), \qquad \Phi_2 = \mathrm{e}^{-B} \wedge \tau(\eta_1 \otimes \bar{\eta}_1^c). \tag{2.45}$$

Therefore, they give a proper SU(2) structure on the manifold, which takes the form (2.35).[15]

To summarize, due to the fact that the pure spinors have definite chirality there is no case which interpolates between the trivial structure and the SU(2) structure case. This can also be understood from the fact that a pair of nowhere vanishing spinors η, η^c spans the space of given chirality. Therefore, all linearly independent spinors have to be of opposite chirality and thus cannot be parallel to η at any point in Y_4. Thus, generic SU(2) × SU(2) structures cannot exist but always have to be SU(2) or trivial structures. Note that our conclusion crucially depends on the assumption that η_1 and η_2 are nowhere vanishing. Therefore the case of a generic warp factor deserves a separate analysis which, however, we do not go into here.

2.4 Moduli space of SU(2) structures

The aim of this section is to determine the moduli space of SU(2) structures. Let us first observe that an eight-dimensional Weyl spinor of SO(4, 4) decomposes under SU(2) × SU(2) as[16]
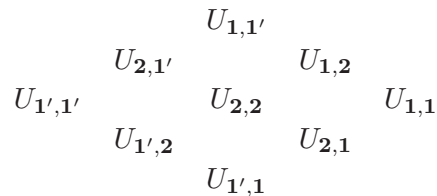
$$\mathbf{8}^s \to (\mathbf{2}, \mathbf{2}) \oplus 4(\mathbf{1}, \mathbf{1}), \qquad \mathbf{8}^c \to 2(\mathbf{2}, \mathbf{1}) \oplus 2(\mathbf{1}, \mathbf{2}). \tag{2.46}$$

Note that, exactly as in (2.8), the two conjugate spinors decompose differently. eq. (A.9) gives a canonical choice for the sign of the chirality operator. Hence, the $\mathbf{8}^s$ ($\mathbf{8}^c$) representation corresponds to forms of even (odd) degree. Let us denote the space of forms transforming in the $(\mathbf{r}, \mathbf{s})$ representation of SU(2) × SU(2) by $U_{\mathbf{r},\mathbf{s}}$. They can be arranged in a diamond as given in table 1, where the prime is used to distinguish the several singlets. In section 2.1 we showed that for a background to have 16 supercharges it is necessary to remove all massive gravitino multiplets which corresponds to projecting out all SU(2) doublets. This eliminates the entire $\mathbf{8}^c$ representation (or equivalently all odd forms in $U_{\mathbf{2},\mathbf{1}'}, U_{\mathbf{1},\mathbf{2}}, U_{\mathbf{1}',\mathbf{2}}, U_{\mathbf{2},\mathbf{1}}$) leaving only the $\mathbf{8}^s$ (i.e. the even forms in table 1). This is consistent with the result of the previous section that backgrounds with 16 supercharges require an SU(2) structure described by pure spinors of positive chirality.

Now we are able to derive the moduli space. For this, let us first discuss the parameter space of one single normalizable pure SO(4, 4) spinor. The two conditions (2.36) and (2.37) have a natural interpretation in the isomorphic picture where Φ is a complex

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[15]Strictly speaking, we can only call this a proper SU(2) structure for geometric compactifications since for non-geometric backgrounds there is globally no projection map $TY \to TY$ such that we can compare the two SU(2) factors. However, we can do this projection locally, and thus may compare both SU(2) structures pointwise. In this sense, we can define proper SU(2) structures even for non-geometric backgrounds.

[16]In section 2.3.2 we showed that for backgrounds with 16 supercharges both SU(2) factors must be the same after projection to the tangent space. However, as long as we stay in the framework of generalized geometry and consider pure SO(4, 4) spinors, these two factors are different. Therefore we do the decomposition for SU(2) × SU(2).

$$
\begin{array}{ccccc}
 & & U_{\mathbf{1,1'}} & & \\
 & U_{\mathbf{2,1'}} & & U_{\mathbf{1,2}} & \\
U_{\mathbf{1',1'}} & & U_{\mathbf{2,2}} & & U_{\mathbf{1,1}} \\
 & U_{\mathbf{1',2}} & & U_{\mathbf{2,1}} & \\
 & & U_{\mathbf{1',1}} & &
\end{array}
$$

Table 1. Generalized $SU(2) \times SU(2)$ diamond.

vector. Equation (2.36) and (2.37) ensure that the real and imaginary part of Φ form a pair of space-like orthogonal vectors. Therefore, Φ is left invariant by the group $SO(2,4)$. From section 2.3 we know that a pure normalizable $SO(4,4)$ spinor breaks the structure group to $SU(2,2)$. Both pictures are consistent with each other since $SU(2,2)$ is just the double cover of $SO(2,4)$. The pure spinor Φ therefore parameterizes the space $SO(4,4)/SO(2,4)$. However, the phase of Φ does not affect the $SU(2,2)$ structure. Hence the actual parameter space of a single pure spinor is

$$
\mathcal{M}_\Phi = \frac{SO(4,4)}{SO(2) \times SO(2,4)} \ . \tag{2.47}
$$

The moduli space of $SU(2)$ structures is more conveniently discussed in terms of the real and imaginary parts of the two spinors Φ_i or in other words in terms of four real vectors $\Psi_a, a = 1, \ldots, 4$ in the space of even forms. Then (2.40) and (2.41) just translate into the conditions

$$
\langle \Psi_a, \Psi_b \rangle = c \, \delta_{ab} \, \mathrm{vol}_4 \, , \tag{2.48}
$$

where c parameterizes the scale of the Ψ_a. The four Ψ_a form the singlet corners in table 1 since they are globally defined and thus must be singlets of the structure group.

In order to understand the signature of the $SU(2) \times SU(2)$ diamond (table 1) we use the generalized Hodge operator $*_B$ defined in (2.32). As we explained in section 2.3.2, the operator $*_B$ leaves the $SU(2) \times SU(2)$ representations invariant and its eigenspaces coincide with the representations. The eigenvalues of $*_B$ are ± 1, and the corresponding eigenspaces are of the same dimension.

The eigenspace with eigenvalue $+1$ is spanned by the four spinors Ψ_a, i.e. by the $SU(2) \times SU(2)$ singlets, which is consistent with (2.48). This can be calculated using the form (2.35) and the fact that the J_i, $i = 1, 2, 3$, defined in (2.12), are self-dual with respect to the standard Hodge operator. Therefore, the orthogonal complement $U_{\mathbf{2,2}}$ is the eigenspace with eigenvalue -1. This shows that a choice of Ψ_a already determines the eigenspaces of the generalized Hodge operator and thus the operator itself. Since the composition of the Mukai pairing with $*_B$ is positive definite, the eigenvalue corresponding to some eigenvector of $*_B$ gives also its signature under the Mukai pairing. Therefore we conclude that the Mukai pairing is positive definite on the $SU(2) \times SU(2)$ singlets and negative definite on $U_{\mathbf{2,2}}$.

Thus, we see that the Ψ_a, which respect the condition (2.48), define a space-like four-dimensional subspace in $\Lambda^{\mathrm{even}} T^* Y_4$ or in other words they parameterize the space

SO(4,4)/SO(4). However, we also need to divide out the rotational SO(4) symmetry among the Ψ_a since it does not change the $SU(2) \times SU(2)$ structure. This leaves as the physical parameter space

$$\mathcal{M}_{\Psi_a} = \frac{SO(4,4)}{SO(4) \times SO(4)} \ . \tag{2.49}$$

So far we have analyzed the deformation space of $SU(2) \times SU(2)$ structures. Let us now discuss its appearance in the low energy effective theory. There are basically two different effective theories one can write down. On the one hand it is possible to rewrite the ten-dimensional supergravity in a form where instead of the ten-dimensional Lorentz group only $SO(1,5) \times SO(4)$ with 16 supercharges is manifest. In this formulation no Kaluza-Klein truncation is performed but instead all fields still carry the full ten-dimensional coordinate dependence of the background (1.1). This is the approach of ref. [34], which we already discussed at the end of section 2.1. In the corresponding effective theory the space (2.49) appears as the target space of the Lorentz-scalar deformations which is consistent with the constraints of the corresponding $d = 6$ supergravity.

Alternatively, one can perform a Kaluza-Klein truncation and only keep the light modes on Y_4. This amounts to a truncation of the space of even forms to a finite subspace $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$ with the assumption that the Mukai pairing is non-degenerate on it [13].[17] Concretely this means that we can expand the four real spinors Ψ_a in a basis of $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$. The generalized Hodge star operator $*_B$, which, via the Ψ_a, is globally defined, splits $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$ into the eigenspaces of $*_B$, i.e.

$$\Lambda_{\text{finite}}^{\text{even}} T^* Y_4 = \Lambda_+^{\text{even}} T^* Y_4 \oplus \Lambda_-^{\text{even}} T^* Y_4 \ , \tag{2.50}$$

where the subscript denotes the eigenvalue with respect to $*_B$. Furthermore, these eigenspaces are orthogonal to each other with respect to the Mukai pairing. $\Lambda_+^{\text{even}} T^* Y_4$ consists of $SU(2) \times SU(2)$ singlets only, and thus over each point it is spanned by the Ψ_a. Therefore, each element of $\Lambda_+^{\text{even}} T^* Y_4$ can be written as a linear combination of the Ψ_a where the coefficients may depend on the base point on Y_4, i.e. are functions on Y_4. However, only constant coefficients survive the Kaluza-Klein truncation, and thus $\Lambda_+^{\text{even}} T^* Y_4$ is spanned by the Ψ_a only and has dimension four.

Now let us turn to the eigenspace $\Lambda_-^{\text{even}} T^* Y_4$. It consists of sections in $U_{\mathbf{2},\mathbf{2}}$ and therefore we cannot make the same argument as for $\Lambda_+^{\text{even}} T^* Y_4$. In contrast to the bundle of $SU(2) \times SU(2)$ singlets, $U_{\mathbf{2},\mathbf{2}}$ might be twisted over the manifold and the dimension of $\Lambda_-^{\text{even}} T^* Y_4$ may differ from four, say $n + 4$. Thus, $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$ is a vector space of signature $(4, n + 4)$. The four spinors Ψ_a satisfy (2.48), and therefore span a four-dimensional space-like subspace in $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$. The parameter space describing these configurations is just $\mathbb{R}_+ \times SO(4, n + 4)/SO(n + 4)$, where the $\mathbb{R}_+$ factor corresponds to the gauge freedom contained in the choice of the parameter c in (2.48).

In order to find the moduli space we still have to remove all gauge redundancies. First of all, the constant factor c is just the rescaling of the pure spinors, and thus is of no physical

---

[17]This just corresponds to the assumption that charge conjugation, i.e. the generalized Hodge star operator, preserves $\Lambda_{\text{finite}}^{\text{even}} T^* Y_4$.

significance. Furthermore, the SO(3) symmetry of the geometric two-forms is promoted to an SO(4)-symmetry between the four real components of the compatible pure spinors. Modding out both redundancies we finally arrive at the moduli space

$$\mathcal{M}_{d=6}^{\text{NS}} \ = \ \frac{\text{SO}(4, n+4)}{\text{SO}(4) \times \text{SO}(n+4)} \times \mathbb{R}_+ \,, \tag{2.51}$$

where we now also included the dilaton via the $\mathbb{R}_+$ factor consistently with $N = 4$ supergravity.

The first factor is a quaternionic space which is related to the fact that the heterotic string compactified on the same Y_4 includes this factor as a subspace. The eight supercharges of the heterotic compactification require that this scalar field space is quaternionic. Correspondingly two different superconformal cones can be constructed from (2.51). In appendix B.1 we discuss the $N = 2$ hyperkähler cone [30, 31], and show that the corresponding hyperkähler potential is determined in terms of the pure spinors and the Mukai pairing. It turns out that imposing purity and compatibility conditions are part of taking the hyperkähler SU(2) quotient [41, 42] of flat space. In appendix B.2 we discuss the flat cone which arises in the corresponding $N = 4$ superconformal theory.

The derivation presented so far did not use the absence of SU(2) doublets. We merely confined our attention to deformations of the two-forms. However, let us note that $d\Psi_a$ is an SU(2) × SU(2) doublet and therefore it cannot correspond to a deformation parameter after projecting out the doublets. At best it can be related to the warp factor, which we so far ignored in the paper. Thus, in the absence of a warp factor and without any doublets we have $d\Psi_a = 0$ and Y_4 has to be a $K3$ manifold. This is consistent with the moduli space (2.51), which for $n = 16$ coincides with the moduli space of K3 manifolds (modulo the $\mathbb{R}_+$ factor). We will return to this issue in [27].

We want to stress that the arguments given below (2.50) can be made for any vector bundle that consists of only singlets under the structure group. Since the structure group does not act on the vector bundle, it must be the trivial bundle and we can give a number of nowhere vanishing sections that form an orthonormal basis at every point. Since these sections are globally defined and nowhere vanishing, they can be associated with objects that define the structure group and therefore they (or locally rescaled versions thereof) survive the Kaluza-Klein truncation. Furthermore, we can conclude that any section that survives the truncation must be just a linear combination of these sections. Therefore, the space of light modes resulting from a Kaluza-Klein truncation on this vector bundle has the same dimension as the bundle itself.

As an example, let us apply this general theorem to the case of geometric deformations of some SU(2) structure. The moduli space over a point was derived in section 2.2 to be (2.13). Again we assume that we can find a consistent Kaluza-Klein truncation to a subspace $\Lambda_{\text{finite}}^2 T^* Y_4$ of $\Lambda^2 T^* Y_4$ on which the wedge product is non-degenerate. Then the Hodge star operator $*$ gives a decomposition

$$\Lambda_{\text{finite}}^2 T^* Y_4 = \Lambda_+^2 T^* Y_4 \oplus \Lambda_-^2 T^* Y_4 \,, \tag{2.52}$$

where $\Lambda_+^2 T^* Y_4$ consists of SU(2) singlets only and therefore is spanned by the J_i. Thus, the signature of $\Lambda_{\text{finite}}^2 T^* Y_4$ is $(3, n + 3)$ for some n, and the J_i parameterize a space-like

three-dimensional subspace in this space. Hence, the geometric moduli space is

$$\mathcal{M}^{\text{geom}}_{d=6} = \frac{\text{SO}(3, n+3)}{\text{SO}(3) \times \text{SO}(n+3)} \times \mathbb{R}_+ \,, \tag{2.53}$$

where we included the $\mathbb{R}_+$ factor for the volume of Y_4.

Note that this argument can already be used to identify the signature of the second cohomology of $K3$. The space of two-forms on $K3$ can be decomposed into eigenspaces of the Hodge operator. Then, the elements of the $+1$ eigenspace are spanned by the Kähler form J and the real and imaginary part of the holomorphic two-form Ω defined via (2.9). All other two-forms in the $+1$ eigenspace must be linear combinations of these three over each point. If such a two-form is closed, the coefficients must be constant, and it is just a linear combination of these three forms in cohomology. Therefore, the $+1$ eigenspace of the Hodge operator in the second cohomology class is exactly three-dimensional and hence the signature of $H^2(K3, \mathbb{R})$ is $(3, 19)$.

2.5 Inclusion of the Ramond-Ramond sector

Generalized geometry is a natural generalization of G-structures since it covers the complete moduli space of the NS-NS sector of string theory. This is due to the fact that the group acting on the generalized tangent bundle coincides with the T-duality group of string theory. However, it is also possible to include the RR sector of Type II string theories by extending the T-duality group to the larger U-duality group which also includes transformations between the NS-NS and the RR sector [43]. To do so, one extends the generalized tangent bundle $\mathcal{T}Y$ to the exceptional generalized tangent bundle [18, 19]. The spin group over this bundle is then the U-duality group which coincides with the (non-compact version of the) exceptional group E_{d+1}. It seems natural that the formalism of pure spinors should extend to the case of exceptional generalized geometry [21].

Let us examine this for the case of SU(2)-structures on Y_4. In this case the U-duality group is $E_{5(5)} = \text{SO}(5,5)$ with the T-duality subgroup being $\text{SO}(4,4)$. Let us first look at the decomposition of the representations of $SO(5,5)$ in terms of its maximal subgroup $\text{SO}(4,4) \times \mathbb{R}_+$. The extra $\mathbb{R}_+$-factor corresponds to shifts of the dilaton. The vector representation of $\text{SO}(5,5)$ decomposes as [44]

$$\mathbf{10} \rightarrow \mathbf{8}^v_0 \oplus \mathbf{1}_{+2} \oplus \mathbf{1}_{-2} \,, \tag{2.54}$$

while for the spinor representation we have

$$\mathbf{16} \rightarrow \mathbf{8}^c_{+1} \oplus \mathbf{8}^s_{-1} \,. \tag{2.55}$$

The subscript denotes the charge of the representation under shifts of the dilaton. Finally, the adjoint of $\text{SO}(5,5)$ decomposes as

$$\mathbf{45} \rightarrow \mathbf{28}_0 \oplus \mathbf{8}^v_{+2} \oplus \mathbf{8}^v_{-2} \oplus \mathbf{1}_0 \,. \tag{2.56}$$

Note that because of SO(4,4) triality, the three $\mathbf{8}$ representations can be interchanged pairwise, which, however, has to be done in all three decompositions simultaneously.

Let us now determine the geometric realizations of these representations. For the T-duality group $SO(4,4)$, the vector representation $\mathbf{8}^v$ is (locally) given in geometrical terms by $TY_4 \oplus T^*Y_4$ and analogously the spinor representations $\mathbf{8}^s$ and $\mathbf{8}^c$ by $\Lambda^{\text{even}}T^*Y_4$ and $\Lambda^{\text{odd}}T^*Y_4$, respectively. However, $SO(4,4)$-triality can interchange the three eight-dimensional bundles $TY_4 \oplus T^*Y_4$, $\Lambda^{\text{even}}T^*Y_4$ and $\Lambda^{\text{odd}}T^*Y_4$. To assign them to the representations in the right way, we note — as explained in [43] — that the NS-NS and RR charges together form the $\mathbf{16}$ representation of $E_{5(5)}$.[18] Since the NS charges come from Kaluza-Klein and winding modes, they live in the geometrical bundle $TY_4 \oplus T^*Y_4$. The RR charges arise from D-branes wrapped on internal cycles and therefore sit in $\Lambda^{\text{even}}T^*Y_4$ for type IIA and $\Lambda^{\text{odd}}T^*Y_4$ for type IIB, respectively. Hence the $\mathbf{16}$ representation corresponds to $TY_4 \oplus T^*Y_4 \oplus \Lambda^{\text{even}}T^*Y_4$ in type IIA and to $TY_4 \oplus T^*Y_4 \oplus \Lambda^{\text{odd}}T^*Y_4$ in type IIB [18]. Consequently, the representation $\mathbf{8}^v$ is associated with $\Lambda^{\text{odd}}T^*Y_4$ in type IIA and $\Lambda^{\text{even}}T^*Y_4$ in type IIB, respectively. Altogether we thus have

$$
\begin{aligned}
\mathbf{10} &= (\Lambda^{\text{odd}}T^*Y_4)_0 \oplus (\mathbb{R})_{+2} \oplus (\mathbb{R})_{-2}\,, \\
\mathbf{16} &= (TY_4 \oplus T^*Y_4)_{+1} \oplus (\Lambda^{\text{even}}T^*Y_4)_{-1}\,, \\
\mathbf{45} &= (so(TY_4 \oplus T^*Y_4))_0 \oplus (\Lambda^{\text{odd}}T^*Y_4)_{+2} \oplus (\Lambda^{\text{odd}}T^*Y_4)_{-2} \oplus (\mathbb{R})_0\,,
\end{aligned}
\tag{2.57}
$$

for type IIA, while in type IIB we have

$$
\begin{aligned}
\mathbf{10} &= (\Lambda^{\text{even}}T^*Y_4)_0 \oplus (\mathbb{R})_{+2} \oplus (\mathbb{R})_{-2}\,, \\
\mathbf{16} &= (TY_4 \oplus T^*Y_4)_{+1} \oplus (\Lambda^{\text{odd}}T^*Y_4)_{-1}\,, \\
\mathbf{45} &= (so(TY_4 \oplus T^*Y_4))_0 \oplus (\Lambda^{\text{even}}T^*Y_4)_{+2} \oplus (\Lambda^{\text{even}}T^*Y_4)_{-2} \oplus (\mathbb{R})_0\,.
\end{aligned}
\tag{2.58}
$$

Here $(so(TY_4 \oplus T^*Y_4))$ denotes the Lie-Algebra of $SO(4,4)$ that acts on $TY_4 \oplus T^*Y_4$. The subscripts give the charges under shifts of the dilaton, which do not have a geometric interpretation. Note that the bundle $\Lambda^{\text{even}}T^*Y_4$ appears in different representations in (2.57) and in (2.58). This shows that the embedding of the pure $SO(4,4)$ spinors $\Phi_1, \Phi_2 \in \Lambda^{\text{even}}T^*Y_4$ has to be different for type IIA and type IIB.

In type IIA backgrounds with 16 supercharges the situation is straightforward. We already argued that in this case we have to project out all $SU(2) \times SU(2)$ doublets or correspondingly $\Lambda^{\text{odd}}T^*Y_4$ together with $TY_4 \oplus T^*Y_4$. Eq. (2.57) then implies that $SO(5,5)$ is broken to $SO(4,4) \times \mathbb{R}_+$ by the projection. This in turn says that all scalar degrees of freedom coming from the RR-sector are projected out together with the massive gravitinos. Of course this conclusion is also reached by direct inspection of the massless type IIA spectrum discussed in section 2.1. This observation also immediately says that the local moduli space is unchanged and given by (2.51), i.e.

$$
\mathcal{M}_{d=6}^{\text{IIA}} \;=\; \mathcal{M}_{d=6}^{\text{NS}}\,.
\tag{2.59}
$$

The analogous discussion in type IIB is slightly more involved. From (2.58) we see that neither the additional generators of $SO(5,5)$ are projected out nor can we embed the pure

---

[18]With charges we mean those solutions which are point-like particles in six dimensions that are charged under the NS and RR vectors. In the NS sector, the charges are formed by the momentum and winding modes of the fundamental string that are charged under $g_{m\mu}$ and $B_{m\mu}$, respectively, while the RR charges descend from ten-dimensional D-brane solutions.

spinors into the spinor representation of SO(5, 5). However, from (2.58) we see that we can embed the SO(4, 4) spinors into the vector representation of SO(5, 5). More precisely, we can either embed the complex pure spinors Φ_1 and Φ_2 into complex SO(5, 5) vectors or, alternatively, use their real and imaginary parts denoted by Ψ_a in the previous section and embed them into real SO(5, 5) vector representations. We use (2.58) to decompose the SO(5, 5) vector into its components

$$\zeta = (\zeta^+, \zeta^s, \tilde{\zeta}^s) \,, \tag{2.60}$$

where ζ^+ lives in $\Lambda^{\mathrm{even}} T^* Y_4$ while $\zeta^s, \tilde{\zeta}^s$ are the two singlets. Then the embedding of the four Ψ_a into SO(5, 5) vectors ζ_a is given by

$$\zeta_a = (\Psi_a, 0, 0) \,, \qquad a = 1, \ldots, 4 \,. \tag{2.61}$$

This results in a set of four orthonormal space-like SO(5, 5) vectors ζ_a which — after modding out the rotational symmetry between them — parameterize the space

$$\mathcal{M}_{\zeta_a} = \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(4) \times \mathrm{SO}(1,5)} \,. \tag{2.62}$$

However, this cannot be the correct parameter space yet. As one can read off from (2.58), the four vectors ζ_a are not charged under the dilaton shift. Thus, the dilaton is not yet included in the parameter space (2.62). Reconsidering (2.54) shows that the two singlets are charged under dilaton shifts, and together form a real SO(1, 1) vector. If we impose a normalization condition on this vector, it parameterizes SO(1, 1) and therefore the dilaton degree of freedom ϕ. We can embed this $SO(1, 1)$ vector into an SO(5, 5) vector ζ_5 using (2.60), i.e.

$$\zeta_5 = \frac{1}{\sqrt{2}} (0, \mathrm{e}^\phi, \mathrm{e}^{-\phi}) \,. \tag{2.63}$$

We see that the ζ_I, $I = 1, \ldots, 5$, are all space-like and satisfy

$$\langle \zeta_I, \zeta_J \rangle_5 = \delta_{IJ} \,, \tag{2.64}$$

where we gauge-fixed the parameter c in (2.48) to be 1.

The stabilizer of this set of vectors is naturally given by SO(5) $\subset$ SO(5, 5) which are the rotations in the space perpendicular to all ζ_I. The generators of this SO(5) are the generators of the SU(2) × SU(2) structure which lie in the T-duality subgroup SO(4, 4) together with the generators of the following transformation

$$\delta_{\beta^{(\mathbf{2},\mathbf{2})}} \;\; \zeta \longmapsto \frac{1}{\sqrt{2}} \left((\mathrm{e}^{-\phi} \zeta^s - \mathrm{e}^\phi \tilde{\zeta}^s) \beta^{(\mathbf{2},\mathbf{2})}, \; \mathrm{e}^\phi \langle \zeta^+, \beta^{(\mathbf{2},\mathbf{2})} \rangle_4, \; -\mathrm{e}^{-\phi} \langle \zeta^+, \beta^{(\mathbf{2},\mathbf{2})} \rangle_4 \right) \tag{2.65}$$

where $\beta^{(\mathbf{2},\mathbf{2})} \in U_{\mathbf{2},\mathbf{2}}$. By a straightforward calculation one can check that these transformations are indeed generators of SO(5, 5) which stabilize the ζ_I. Since $\beta^{(\mathbf{2},\mathbf{2})}$ also transforms as an SO(4) $\equiv$ SU(2) × SU(2) vector, the transformations (2.65) together with the generators of SO(4) form the adjoint of SO(5). Therefore, the ζ_I, $I = 1, \ldots, 5$, obeying (2.64), span the space SO(5, 5)/SO(5).

The embedding of the NS sector into $SO(5,5)$ given in (2.61) and (2.63) is not yet generic. Consider the following $SO(5,5)$ transformations that are not part of $SO(4,4) \times SO(1,1)$ and that are not of the form (2.65):[19]

$$\delta_{\alpha^{(2,2)}} \ \zeta \longmapsto \frac{1}{\sqrt{2}} \left(-(e^{-\phi}\zeta^s + e^{\phi}\tilde{\zeta}^s)\alpha^{(2,2)}, \ e^{\phi}\langle\zeta^+, \alpha^{(2,2)}\rangle_4, \ e^{-\phi}\langle\zeta^+, \alpha^{(2,2)}\rangle_4 \right),$$

$$\delta_{\alpha^{(1,1)}} \ \zeta \longmapsto \frac{1}{\sqrt{2}} \left(-(e^{-\phi}\zeta^s + e^{\phi}\tilde{\zeta}^s)\alpha^{(1,1)}, \ e^{\phi}\langle\zeta^+, \alpha^{(1,1)}\rangle_4, \ e^{-\phi}\langle\zeta^+, \alpha^{(1,1)}\rangle_4 \right), \qquad (2.66)$$

$$\delta_{\beta^{(1,1)}} \ \zeta \longmapsto \frac{1}{\sqrt{2}} \left((e^{-\phi}\zeta^s - e^{\phi}\tilde{\zeta}^s)\beta^{(1,1)}, \ e^{\phi}\langle\zeta^+, \beta^{(1,1)}\rangle_4, \ -e^{-\phi}\langle\zeta^+, \beta^{(1,1)}\rangle_4 \right),$$

where the superscript of the deformation parameter denotes in which $SU(2) \times SU(2)$ representation the spinor parameter lies. These transformations modify the embedding and therefore introduce additional degrees of freedom. As we will see shortly they correspond to the RR-scalars. However, let us first observe that $\delta_{\alpha^{(1,1)}}$ rotates ζ_5 and the ζ_a, $a = 1, \ldots, 4$, into each other. Therefore, $\alpha^{(1,1)}$ parameterizes gauge degrees of freedom which enhance the $SO(4)$ symmetry between the ζ_a to an $SO(5)$ symmetry between the ζ_I. The remaining parameters $\alpha^{(2,2)}$ and $\beta^{(1,1)}$ genuinely modify the embedding and therefore correspond to additional physical scalar degrees of freedom. They are precisely the RR-scalars of type IIB. To see this, we split the formal sum of RR-fields

$$C = C_0 + C_2 + C_4 \ \in \Lambda^{\text{even}} T^* Y_4, \qquad (2.67)$$

into $SU(2) \times SU(2)$ representations and associate them with the parameters, i.e.

$$C = C^{(2,2)} + C^{(1,1)} \equiv \alpha^{(2,2)} + \beta^{(1,1)}. \qquad (2.68)$$

Note that we only considered the infinitesimal transformations in (2.66). Therefore a finite shift in the C-fields corresponds to the exponentiation of (2.66).

The previous discussion shows that in type IIB apart from the pure spinors Ψ_a also the dilaton and the RR scalars C are part of the moduli space and therefore the space given in (2.62) has to be modified. We just argued that the basic objects are five $SO(5,5)$ vectors ζ_I that satisfy (2.64) and which are stabilized by $SO(5)$. In addition there is an $SO(5)$ gauge symmetry rotating the vectors into each other. Therefore the physical parameter space is

$$\mathcal{M}_{\zeta_I} \ = \ \frac{SO(5,5)}{SO(5) \times SO(5)}. \qquad (2.69)$$

Finally, let us derive the moduli space of type IIB $SU(2)$-structure backgrounds in six dimensions. As in section 2.4 we assume that a consistent Kaluza-Klein truncation which leaves the Mukai-pairing non-degenerate exists. By repeating the argument given at the end of section 2.4 we conclude that the subspace of positive signature is globally spanned by the ζ_I and thus has dimension five. Therefore, the ζ_I span a space-like five-dimensional subspace in a vector space of signature $(5, n+5)$. Hence, the moduli space is

$$\mathcal{M}_{d=6}^{\text{IIB}} \ = \ \frac{SO(5, n+5)}{SO(5) \times SO(n+5)}. \qquad (2.70)$$

---

[19] Note that the specific parameterization chosen depends on the dilaton value and the decomposition of $\Lambda^{\text{even}} T^* Y_4$ into the $(1,1)$ and $(2,2)$ representations.

This result is consistent with the corresponding chiral supergravity in $d = 6$ [33].

Both $\mathcal{M}_{d=6}^{\mathrm{IIA}}$ and $\mathcal{M}_{d=6}^{\mathrm{IIB}}$ are the base of a flat cone corresponding to the associated superconformal supergravity. This is further discussed in appendix B.2.

This concludes our analysis of type II compactified on an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure manifold Y_4 of dimension four. Let us now turn to compactifications on six-dimensional manifolds Y_6.

3   Compactifications on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure manifolds in four space-time dimensions

So far we discussed backgrounds with six space-time dimensions and 16 supercharges. Let us now study backgrounds of the form $M_{3,1} \times Y_6$ but with the same number of supercharges. Thus, in this section we focus on six-dimensional manifolds Y_6 which have $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure.

To be more precise, we consider backgrounds whose Lorentz group is $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$. The ten-dimensional spinor representation decomposes accordingly as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}), \tag{3.1}$$

where the $\mathbf{2}$ is a Weyl spinor of $\mathrm{SO}(1,3)$ while the $\mathbf{4}$ denotes the spinor representation of $\mathrm{SO}(6)$.

Backgrounds allowing for sixteen supercharges must admit two or four globally defined spinors, which corresponds to manifolds Y_6 with a reduced structure group $\mathrm{SU}(2)$ or $\mathrm{SU}(2) \times \mathrm{SU}(2)$, respectively. Similar to section 2, we start with a general analysis of the spectrum of type II supergravities in such backgrounds, and then determine their moduli space explicitly by use of the pure spinor formalism of generalized geometry.

3.1   Field decomposition for $d = 4$

We start by analyzing the massless type II supergravity fields in terms of their representations under the four-dimensional Lorentz group and the structure group and show analogously to section 2.1 how they assemble in $N = 4$ multiplets, in the spirit of [13].

Again, we use the light-cone gauge and only consider representations of $\mathrm{SO}(2)$ instead of the whole $\mathrm{SO}(1,3)$ Lorentz group. Since we treat the case of an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure group, we examine the decomposition of massless type II supergravity fields under the group $\mathrm{SO}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. For this, let us recall the decomposition of the two Majorana-Weyl representations $\mathbf{8}^s$ and $\mathbf{8}^c$ and the vector representation $\mathbf{8}^v$ under the breaking $\mathrm{SO}(8) \to \mathrm{SO}(2) \times \mathrm{SO}(6) \to \mathrm{SO}(2) \times \mathrm{SU}(2)$. We get

$$\begin{aligned}
\mathbf{8}^s &\to \mathbf{4}_{+\frac{1}{2}} \oplus \bar{\mathbf{4}}_{-\frac{1}{2}} \to 2\,\mathbf{1}_{\pm\frac{1}{2}} \oplus 2\,\mathbf{2}_{\pm\frac{1}{2}}, \\
\mathbf{8}^c &\to \mathbf{4}_{-\frac{1}{2}} \oplus \bar{\mathbf{4}}_{+\frac{1}{2}} \to 2\,\mathbf{1}_{\pm\frac{1}{2}} \oplus 2\,\mathbf{2}_{\pm\frac{1}{2}}, \\
\mathbf{8}^v &\to \mathbf{1}_{\pm 1} \oplus \mathbf{6}_0 \to \mathbf{1}_{\pm 1} \oplus 2\,\mathbf{1}_0 \oplus \mathbf{2}_0 \oplus \bar{\mathbf{2}}_0,
\end{aligned} \tag{3.2}$$

where the subscript denotes the $\mathrm{SO}(2)$ charge, i.e. the helicity. We note that both Majorana-Weyl representations decompose in the same way under the $\mathrm{SU}(2)$ structure

group. Therefore, we expect type IIA and type IIB compactifications to give the same theory in $d = 4$.

In type IIA the massless fermionic degrees of freedom originate from the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^c)$ representation of $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R$, while in type IIB they form the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^s)$ representation. Under the decomposition $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R \to \mathrm{SO}(2) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ they split as

$$(\mathbf{8}^s, \mathbf{8}^v) \to 2(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus 4(\mathbf{1}, \mathbf{1})_{\pm\frac{1}{2}} \oplus 4(\mathbf{1}, \mathbf{2})_{\pm\frac{1}{2}} \oplus (\mathbf{2}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus 2(\mathbf{2}, \mathbf{1})_{\pm\frac{1}{2}} \oplus 2(\mathbf{2}, \mathbf{2})_{\pm\frac{1}{2}} \,,$$

$$(\mathbf{8}^v, \mathbf{8}^c) \to 2(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus 4(\mathbf{1}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus 2(\mathbf{1}, \mathbf{2})_{\pm\frac{1}{2}} \oplus 4(\mathbf{2}, \mathbf{1})_{\pm\frac{1}{2}} \oplus 2(\mathbf{2}, \mathbf{2})_{\pm\frac{1}{2}} \,,$$
$$\tag{3.3}$$

with $(\mathbf{8}^v, \mathbf{8}^s)$ decomposing like the $(\mathbf{8}^v, \mathbf{8}^c)$ due to (3.2). We see that half of the gravitinos, come in the $(\mathbf{1}, \mathbf{1})$ representation while the other half is in the doublet representations $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{1})$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The latter ones again correspond to massive gravitino multiplets that must be projected out to end up with standard $N = 4$ supergravity. After this projection, the fermionic components in the $(\mathbf{1}, \mathbf{1})$ become part of the gravity multiplet, while the $(\mathbf{2}, \mathbf{2})$ components correspond to the fermionic degrees of freedom in the $N = 4$ vector multiplets.

The bosonic fields can be treated in the same way. In the NS-sector we decompose the $(\mathbf{8}^v, \mathbf{8}^v)$ which corresponds to $E_{MN} = g_{MN} + B_{MN} + \phi \eta_{MN}$ as

$$\begin{aligned}
E_{\mu\nu} &: (\mathbf{1}, \mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1}, \mathbf{1})_{\mathbf{T}} \oplus (\mathbf{1}, \mathbf{1})_{\mathbf{0}} \,, \\
E_{\mu m} &: 2(\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}} \oplus 2(\mathbf{1}, \mathbf{2})_{\pm\mathbf{1}} \,, \\
E_{m\mu} &: 2(\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}} \oplus 2(\mathbf{2}, \mathbf{1})_{\pm\mathbf{1}} \,, \\
E_{mn} &: 4(\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus 4(\mathbf{1}, \mathbf{2})_{\mathbf{0}} \oplus 4(\mathbf{2}, \mathbf{1})_{\mathbf{0}} \oplus 4(\mathbf{2}, \mathbf{2})_{\mathbf{0}} \,,
\end{aligned} \tag{3.4}$$

where $\mathbf{T}$ denotes the antisymmetric tensor, and the singlet $(\mathbf{1}, \mathbf{1})_{\mathbf{0}}$ corresponds to the four-dimensional dilaton ϕ. After the projection we are left with the four-dimensional metric, the antisymmetric two-tensor, four vectors and five scalars $E_{mn}^{(\mathbf{1}, \mathbf{1})}$ and ϕ all in the $(\mathbf{1}, \mathbf{1})$ representation. In addition we also have four scalars $E_{mn}^{(\mathbf{2}, \mathbf{2})}$ in the $(\mathbf{2}, \mathbf{2})$ representation. As in $d = 6$, they can be associated with deformations of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure background.

In the RR-sector we need to consider the decomposition of $(\mathbf{8}^s, \mathbf{8}^c)$ for type IIA and of $(\mathbf{8}^s, \mathbf{8}^s)$ for type IIB. They both decompose similarly as

$$\begin{aligned}
(\mathbf{8}^s, \mathbf{8}^c) \to\, &4(\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}} \oplus 8(\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus 2(\mathbf{1}, \mathbf{2})_{\pm\mathbf{1}} \oplus 4(\mathbf{1}, \mathbf{2})_{\mathbf{0}} \\
&\oplus 2(\mathbf{2}, \mathbf{1})_{\pm\mathbf{1}} \oplus 4(\mathbf{2}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{2}, \mathbf{2})_{\pm\mathbf{1}} \oplus 2(\mathbf{2}, \mathbf{2})_{\mathbf{0}} \,.
\end{aligned} \tag{3.5}$$

Projecting out all $\mathrm{SU}(2) \times \mathrm{SU}(2)$ doublets leaves us with four vectors and eight scalars in the $(\mathbf{1}, \mathbf{1})$ representation and one vector and two scalars in the $(\mathbf{2}, \mathbf{2})$ representation.

Together these fields can be arranged into an $N = 4$ gravity multiplet plus three $N = 4$ vector multiplets. The gravity multiplet contains the graviton, four gravitini, six vector fields, four Weyl fermions and two scalars all in the $(\mathbf{1}, \mathbf{1})$ representation. The vector multiplets each contain one vector, four gaugini and six scalars. Two of them are also in the $(\mathbf{1}, \mathbf{1})$ representation while the third vector multiplet is in the $(\mathbf{2}, \mathbf{2})$ representation. We see that, in contrast to $d = 6$, not all fields in the $(\mathbf{1}, \mathbf{1})$ representation are part of the

gravity multiplet but they also form two vector multiplets. This corresponds to the fact that the six-dimensional gravity multiplet reduces to a four-dimensional gravity multiplet plus two vector multiplets.

As we already discussed in section 2.1, these multiplets still consist of ten-dimensional fields that are reordered in such a way that they form $N = 4$ multiplets. In the corresponding rewriting of the action only $SO(1,3) \times SO(6)$ symmetry and $N = 4$ supersymmetry are manifest. Then we projected out the $SU(2) \times SU(2)$ doublets to achieve a theory that actually allows only for $N = 4$ supersymmetry.

3.2 Y_6 with $SU(2)$ structure

A compact six-dimensional manifold Y_6 has a structure group G that is a subgroup of $SO(6)$, which acts on spinors via its double cover $SU(4)$. One globally defined and nowhere vanishing spinor η on Y_6 defines an $SU(3)$ structure [5, 7]. This is due to the fact that the structure group G must admit a singlet and this requires $SU(4)$ to be broken to a subgroup contained in $SU(3)$. Therefore, the existence of η implies $G \subset SU(3)$.

If we assume that there are two such spinors η_1 and η_2 that are orthogonal at each point, the structure group is broken further. Now there must be two singlets of the structure group and therefore, G is contained in $SU(2)$. The spinors η_1 and η_2 each define an $SU(3)$ structure which intersect in an $SU(2)$ structure. The two distinct $SU(3)$-structures are defined by [7, 22]

$$
\begin{aligned}
J_{mn}^{(1)} &:= -\,\mathrm{i}\,\bar{\eta}_1\gamma_{mn}\eta_1\,, & \Omega_{mnp}^{(1)} &:= \mathrm{i}\,\bar{\eta}_1^c\gamma_{mnp}\eta_1\,, & m,n,p &= 1,\ldots,6\,, \\
J_{mn}^{(2)} &:= -\,\mathrm{i}\,\bar{\eta}_2\gamma_{mn}\eta_2\,, & \Omega_{mnp}^{(2)} &:= \mathrm{i}\,\bar{\eta}_2^c\gamma_{mnp}\eta_2\,,
\end{aligned}
\tag{3.6}
$$

where our spinor convention are summarized in appendix A. With the use of the Fierz identities (A.4) one can express them in terms of an $SU(2)$ structure:

$$
\begin{aligned}
J^{(1)} &= J + \frac{\mathrm{i}}{2}K \wedge \bar{K}\,, & \Omega^{(1)} &= \Omega \wedge K\,, \\
J^{(2)} &= J - \frac{\mathrm{i}}{2}K \wedge \bar{K}\,, & \Omega^{(2)} &= \Omega \wedge \bar{K}\,.
\end{aligned}
\tag{3.7}
$$

The $SU(2)$ structure is defined by the complex one-form [7, 22]

$$
K_m := \bar{\eta}_2^c\gamma_m\eta_1\,,
\tag{3.8}
$$

and the two-forms J and Ω given by

$$
J_{mn} = -\frac{1}{2}\,\mathrm{i}\,(\bar{\eta}_1\gamma_{mn}\eta_1 + \bar{\eta}_2\gamma_{mn}\eta_2)\,, \qquad \Omega_{mn} = \mathrm{i}\,\bar{\eta}_2\gamma_{mn}\eta_1\,.
\tag{3.9}
$$

J and Ω fulfill (2.10), while K satisfies

$$
K_mK^m = 0\,, \qquad \bar{K}_mK^m = 2\,, \qquad \iota_K J = 0\,, \qquad \iota_K\Omega = \iota_{\bar{K}}\Omega = 0\,.
\tag{3.10}
$$

K also specifies an almost product structure

$$
P_m{}^n := K_m\bar{K}^n + \bar{K}_mK^n - \delta_m{}^n\,,
\tag{3.11}
$$

in that

$$P_m{}^n P_n{}^p = \delta_m{}^p \ . \tag{3.12}$$

As can be seen from (3.7), this almost product structure is related to the almost complex structures $J^{(i)}$ of the two SU(3) structures by

$$P_m{}^n = -J^{(1)}{}_m{}^p \, J^{(2)}{}_p{}^n \ . \tag{3.13}$$

From (3.10) we can see that K_m and $\bar{K}_m$ are both eigenvectors of $P_m{}^n$ with eigenvalue $+1$. The vectors orthogonal to $K_m, \bar{K}_m$ have eigenvalue -1 as can be seen from (3.11). Therefore, K_m and $\bar{K}_m$ even span the $+1$ eigenspace.

In terms of stable forms, an SU(2)-structure on Y_6 can be defined by a global complex one-form[20] K which breaks the structure group SO(6) to SO(4) and — as on Y_4 — by three stable two-forms J_i that reduce this group further to SU(2). To assure this breaking of the structure group, all of these forms have to be compatible with each other in that they satisfy (2.11) and (3.10).

Actually, an almost product structure $P_m{}^n$ that has a positive eigenspace of dimension two and a globally defined, nowhere vanishing spinor η are enough to define an SU(2) structure on a manifold of dimension six. The reason is that $P_m{}^n$ reduces the structure group to $SO(2) \times SO(4)$ and therefore also reduces the SU(3) structure defined by η to an SU(2) structure. This fits nicely with the fact that the existence of $P_m{}^n$ is already enough to assure that the quantities defined in (3.6) are of the form (3.7) and thereby indeed define an SU(2) structure on the manifold. Correspondingly, the two globally defined spinors that reduce the structure group to SU(2) are η and $(v_m \gamma^m \eta^c)$ with v_m is any (real) $+1$-eigenvector of P.

Now let us derive the parameter space of SU(2) structures. As before, we have to ensure that we compactify to $N = 4$, and therefore project out all SU(2) doublets, as explained in section 3.1. As we show in appendix C this projection forces the almost product structure P to be rigid. Therefore, the parameter space splits into a part for the two-dimensional identity structure and one for deformations of the SU(2) structure in the four-dimensional subspace. The former is parameterized by K, the latter one by J and Ω. The local parameter space of the SU(2) structure part was already derived in section 2.2 and is given by (2.13). The identity structure is parameterized by the complex one-form K in a two-dimensional space. Its length corresponds to $K \wedge \bar{K}$ and parameterizes the volume of the two-dimensional space. The group $SU(1,1) \equiv Sl(2,\mathbb{R})$ leaves $K \wedge \bar{K}$ invariant, while it acts freely on K. Therefore, its action parameterizes the remaining freedom in choosing K. Since the phase of K is of no relevance, we have to mod out this degree of freedom, and end up with the parameter space $Sl(2,\mathbb{R})/SO(2)$. Therefore, after including the degree of freedom that correspond to the volume of the four-dimensional subspace, we end up with the parameter space

$$\mathcal{M}_{K,J_i} = \frac{SO(3,3)}{SO(3) \times SO(3)} \times \mathbb{R}_+ \times \frac{Sl(2,\mathbb{R})}{SO(2)} \times \mathbb{R}_+ \ . \tag{3.14}$$

---

[20]Note that every one-form is stable by definition.

By the argument we presented in section 2.4, all subspaces which consist of SU(2) singlets remain unchanged when we perform a Kaluza-Klein truncation. Hence, as in section 2.4 only the SU(2) structure parameter space can change and the moduli space is

$$\mathcal{M}_{d=4}^{\text{geom}} = \frac{\text{SO}(3, 3+n)}{\text{SO}(3) \times \text{SO}(3+n)} \times \mathbb{R}_+ \times \frac{Sl(2, \mathbb{R})}{\text{SO}(2)} \times \mathbb{R}_+ \ . \tag{3.15}$$

3.3 Generalized geometry and $\text{SU}(2) \times \text{SU}(2)$ structures on Y_6

As in section 2.3 we now generalize the previous discussion to the case of $\text{SU}(2) \times \text{SU}(2)$ structures using pure $\text{SO}(6, 6)$ spinors. In six dimensions, the condition for the existence of pure spinors has been analyzed in great detail by Hitchin [6]. The result is that a normalizable pure $\text{SO}(6, 6)$ spinor Φ is in one-to-one correspondence to a real stable $\text{SO}(6, 6)$ spinor and hence looses half of its degrees of freedom.

As we already noted in (2.23) two normalizable pure $\text{SO}(6, 6)$ spinors Φ^+ and Φ^- are compatible if and only if [9]

$$\dim L_{\Phi^+} \cap L_{\Phi^-} = 3 \ . \tag{3.16}$$

Eq. (2.24) then implies that $\Phi^\pm$ must be of opposite chirality. From (2.24) we know that for two spinors $\Phi^\pm$ of opposite chirality the annihilator spaces $L_{\Phi^\pm}$ intersect in an odd-dimensional space. Thus, eq. (3.16) can be understood as telling us that $L_{\Phi^+} \cap L_{\Phi^-}$ must be neither one- nor five-dimensional. From (2.21) one can deduce [12, 13]

$$\langle \Phi^+, \Gamma^M \Phi^- \rangle = \langle \bar{\Phi}^+, \Gamma^M \Phi^- \rangle = 0, \tag{3.17}$$

where Γ^M, $M = 1, \ldots, 12$, are gamma-matrices of $\text{SO}(6, 6)$. This is a more convenient form for the compatibility condition. In addition, we can impose the normalization condition

$$\langle \Phi^+, \bar{\Phi}^+ \rangle = \langle \Phi^-, \bar{\Phi}^- \rangle \ . \tag{3.18}$$

So far, (3.17) and (3.18) only define an $\text{SU}(3) \times \text{SU}(3)$ structure on Y_6. In order to construct an $\text{SU}(2) \times \text{SU}(2)$ structure, one has to introduce further objects that are globally defined and compatible with the spinors introduced so far. One way to proceed is by mimicking the SU(2) structure construction and define two $\text{SU}(3) \times \text{SU}(3)$ structures with compatibility conditions imposed such that they intersect in an $\text{SU}(2) \times \text{SU}(2)$ structure. Each $\text{SU}(3) \times \text{SU}(3)$ structure already defines a generalized metric on $TY_6 \oplus T^*Y_6$, and these two generalized metrics must coincide in a well-defined string background. We could express each $\text{SU}(3) \times \text{SU}(3)$ structure in terms of a pair of compatible pure spinors $\Phi_{(i)}^\pm$, $i = 1, 2$, and formulate conditions on these pure spinors to ensure that we end up with an $\text{SU}(2) \times \text{SU}(2)$ structure. However, it is more elegant to start with four generalized almost complex structures $\mathcal{J}_{\Phi_{(i)}^\pm}$ (cf. section 2.3) and switch to the notion of pure spinors later.

First of all, to ensure that we are able to diagonalize all four $\mathcal{J}_{\Phi_{(i)}^\pm}$ simultaneously, we demand that they commute with each other. This implies that we can decompose $(TY_6 \oplus T^*Y_6)_{\mathbb{C}}$ in the way of (2.17) for each pair $\mathcal{J}_{\Phi_{(i)}^\pm}$, $i = 1, 2$, simultaneously. Each of the four components in (2.17) is of complex dimension 3, and one of the SU(3) factors of the $\text{SU}(3) \times \text{SU}(3)$ structure group acts on L_{++} and L_{--} while the other one acts on L_{+-} and

L_{-+}. We want to define both pairs in such a way that together they break the structure group to $SU(2) \times SU(2)$. For this, the complex three-dimensional $SU(3)$ representation L_{++} must split into a two- and a one-dimensional part. The same must hold for the other components in (2.17). This corresponds to

$$\dim_{\mathbb{C}} \left(L_{++}^{(1)} \cap L_{--}^{(2)} \right) = 2 = \dim_{\mathbb{C}} \left(L_{--}^{(1)} \cap L_{++}^{(2)} \right) ,$$
$$\dim_{\mathbb{C}} \left(L_{+-}^{(1)} \cap L_{-+}^{(2)} \right) = 2 = \dim_{\mathbb{C}} \left(L_{-+}^{(1)} \cap L_{+-}^{(2)} \right) . \tag{3.19}$$

Note that the first equation implies the second and the third one implies the fourth.

Analogously to (3.13), we can also define two generalized almost product structures by

$$\mathcal{P}_\pm = -\mathcal{J}_{\Phi_{(1)}^\pm} \mathcal{J}_{\Phi_{(2)}^\pm} , \tag{3.20}$$

which satisfy

$$\mathcal{P}_\pm^2 = 1 . \tag{3.21}$$

In order to determine the dimension of the eigenspaces let us observe that (3.19) implies

$$\dim_{\mathbb{C}} L_{\Phi_{(1)}^\pm} \cap L_{\bar{\Phi}_{(2)}^\pm} = \dim_{\mathbb{C}} L_{\bar{\Phi}_{(1)}^\pm} \cap L_{\Phi_{(2)}^\pm} = 4 , \tag{3.22}$$

where we used the relations

$$L_{\Phi_{(i)}^+} = L_{++}^{(i)} \oplus L_{+-}^{(i)} , \qquad L_{\Phi_{(i)}^-} = L_{++}^{(i)} \oplus L_{-+}^{(i)} , \qquad i = 1, 2 . \tag{3.23}$$

Therefore, both $\mathcal{P}_+$ and $\mathcal{P}_-$ have an eigenspace of dimension eight for the eigenvalue -1, and correspondingly, an eigenspace of dimension four for the eigenvalue $+1$.

Let us now show that the $\mathcal{P}_\pm$ are identical. To see this we recall that both pairs of pure spinor must define the same generalized metric $\mathcal{G}$ on $TY_6 \oplus T^*Y_6$, i.e.

$$\mathcal{G} = \mathcal{I} \mathcal{J}_{\Phi_{(1)}^+} \mathcal{J}_{\Phi_{(1)}^-} = \mathcal{I} \mathcal{J}_{\Phi_{(2)}^+} \mathcal{J}_{\Phi_{(2)}^-} , \tag{3.24}$$

where $\mathcal{I}$ is the bilinear form of split signature that is induced by the canonical pairing of tangent and cotangent space. Therefore, $\mathcal{J}_{\Phi_{(2)}^+}$ is already determined by the other pure spinors, via the equation

$$\mathcal{J}_{\Phi_{(2)}^+} = -\mathcal{J}_{\Phi_{(1)}^+} \mathcal{J}_{\Phi_{(1)}^-} \mathcal{J}_{\Phi_{(2)}^-} . \tag{3.25}$$

This implies

$$\mathcal{P}^+ = -\mathcal{J}_{\Phi_{(1)}^+} \mathcal{J}_{\Phi_{(2)}^+} = \mathcal{J}_{\Phi_{(1)}^+} \mathcal{J}_{\Phi_{(1)}^+} \mathcal{J}_{\Phi_{(1)}^-} \mathcal{J}_{\Phi_{(2)}^-} = -\mathcal{J}_{\Phi_{(1)}^-} \mathcal{J}_{\Phi_{(2)}^-} = \mathcal{P}^- . \tag{3.26}$$

Therefore we see that an $SU(2) \times SU(2)$ structure can alternatively be defined by a pair of compatible pure spinors $\Phi_{(1)}^+$, $\Phi_{(1)}^-$ and a generalized almost product structure $\mathcal{P}$ which has the following properties:

1. $\mathcal{P}^2 = 1$.

2. $\mathcal{P}$ is symmetric with respect to $\mathcal{I}$.

3. $\mathcal{P}$ commutes with the generalized almost complex structures $\mathcal{J}_{\Phi_{(1)}^{\pm}}$.

4. The eigenspaces of $\mathcal{P}$ to the eigenvalues -1 and $+1$ are of dimension 8 and 4, respectively.

These conditions replace (3.19). Note that the second and third conditions ensure that $\mathcal{P}$ is also symmetric with respect to the metric defined by $\mathcal{J}_{\Phi_{(1)}^{+}}$ and $\mathcal{J}_{\Phi_{(1)}^{-}}$.

Since the generalized almost complex structures $\mathcal{J}_{\Phi_{(i)}^{-}}$ are skew-symmetric with respect to the canonical pairing and commute with each other, $\mathcal{P}$ is indeed symmetric with respect to the canonical pairing by construction. This implies that the canonical pairing is block-diagonal with respect to the splitting of the bundle induced by $\mathcal{P}$. Therefore, $\mathcal{P}$ reduces the structure group to $\mathrm{SO}(4,4) \times \mathrm{SO}(2,2)$. Since it commutes with $\mathcal{J}_{\Phi_{(1)}^{+}}$ and $\mathcal{J}_{\Phi_{(1)}^{-}}$, both generalized almost complex structures are similarly block-diagonal with respect to this splitting.

Thus, we conclude that reducing an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure to some $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure corresponds to the fact that one is able to globally split $TY_6 \oplus T^*Y_6$ into

$$TY_6 \oplus T^*Y_6 = (T_2 Y_6 \oplus T_2^* Y_6) \oplus (T_4 Y_6 \oplus T_4^* Y_6), \qquad (3.27)$$

where $T_4 Y_6 \oplus T_4^* Y_6$ is the eight-dimensional vector bundle that is the -1 eigenspace of $\mathcal{P}$ at every point, and $T_2 Y_6 \oplus T_2^* Y_6$ is correspondingly the four-dimensional vector bundle that forms the $+1$ eigenspace of $\mathcal{P}$ at every point.[21] The pure spinor pair $\Phi_{(1)}^{\pm}$, corresponding to $\mathcal{J}_{\Phi_{(1)}^{\pm}}$, defines some $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure on $T_4 Y_6 \oplus T_4^* Y_6$ and some trivial structure on $T_2 Y_6 \oplus T_2^* Y_6$, i.e. $T_2 Y_6 \oplus T_2^* Y_6$ is the trivial bundle. On $T_4 Y_6 \oplus T_4^* Y_6$, we can redo the analysis of section 2 since the dimension of $T_4 Y_6 \oplus T_4^* Y_6$ is eight.

Let us make this structure slightly more explicit by considering the pure spinors $\Phi^{\pm}$ that correspond to $\mathcal{J}_{\Phi^{\pm}}$.[22] First, let us fix the generalized almost product structure $\mathcal{P}$ and investigate the structure of Φ^+ and Φ^-. Eq. (3.27) induces a splitting of the $\mathrm{SO}(6,6)$ spinor space $\Lambda^{\bullet} T^* Y_6$, i.e.

$$\Lambda^{\bullet} T^* Y_6 = \Lambda^{\bullet} T_2^* Y_6 \wedge \Lambda^{\bullet} T_4^* Y_6, \qquad (3.28)$$

where $\Lambda^{\bullet} T_2^* Y_6$ and $\Lambda^{\bullet} T_4^* Y_6$ are the $\mathrm{SO}(2,2)$ and the $\mathrm{SO}(4,4)$ spinor bundle over Y_6, respectively. This decomposition carries over to the chiral subbundles

$$\begin{aligned}
\Lambda^{\mathrm{even}} T^* Y_6 &= \Lambda^{\mathrm{even}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6 \ \oplus\ \Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{odd}} T_4^* Y_6, \\
\Lambda^{\mathrm{odd}} T^* Y_6 &= \Lambda^{\mathrm{even}} T_2^* Y_6 \wedge \Lambda^{\mathrm{odd}} T_4^* Y_6 \ \oplus\ \Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6.
\end{aligned} \qquad (3.29)$$

The direct sum on the right-hand side holds globally, since, by use of $\mathcal{P}$, we can define chirality operators for $\Lambda^{\bullet} T_2^* Y_6$ and $\Lambda^{\bullet} T_4^* Y_6$ independently. In other words, the structure group does not mix the spinor bundles $\Lambda^{\mathrm{even}} T_4^* Y_6$ and $\Lambda^{\mathrm{odd}} T_4^* Y_6$ and the spinor bundles $\Lambda^{\mathrm{even}} T_2^* Y_6$ and $\Lambda^{\mathrm{odd}} T_2^* Y_6$.

---

[21]Properly written, (3.27) reads $\mathcal{T} Y_6 = \mathcal{T}_2 Y_6 \oplus \mathcal{T}_4 Y_6$.

[22]Here and in the following, we drop the subscripts of $\Phi_{(1)}^+$ and $\Phi_{(1)}^-$.

Moreover, since the generalized almost complex structures commute with $\mathcal{P}$, they split under (3.27) into a generalized almost complex structure on each component. Correspondingly, using (3.29), the pure spinors Φ^+ and Φ^- globally decompose into pure spinors on the spinor subbundles. As already argued above, the spinor bundles on the right-hand side of eq. (3.29) do not mix under the action of the structure group, and therefore, the components of Φ^+ and Φ^- on the subbundles can be analyzed separately. Their components on $\Lambda^\bullet T_4^* Y_6$ must define an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure. However, we already discussed the case of an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure group on some vector bundle of dimension eight in section 2. We know from section 2.3 that an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure group on $T_4 Y_6 \oplus T_4^* Y_6$ is already defined by two pure spinors that must have the same chirality. Any additional nowhere vanishing, globally defined pure spinor would break the structure group further. Thus, we can distinguish two cases: Either both spinor components on $\Lambda^\bullet T_4^* Y_6$ lie in $\Lambda^{\mathrm{odd}} T_4^* Y_6$ or in $\Lambda^{\mathrm{even}} T_4^* Y_6$. Note that in both cases we are left with two pure spinors of opposite chirality in $\Lambda^\bullet T_2^* Y_6$ which define a trivial structure on $T_2 Y_6 \oplus T_2^* Y_6$.

In the first case, both pure spinors on $T_4 Y_6 \oplus T_4^* Y_6$ are of negative chirality. As we showed in section 2.3, these two pure spinors define an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure where the two $\mathrm{SU}(2)$ factors have trivial intersection. Thus Y_6 admits a trivial structure, i.e. is parallelizable, and the background has 32 supercharges. As in section 2.3, we do not discuss this case any further.

The second possibility is that both spinor components are of positive chirality and define — analogously to section 2.3 — a proper $\mathrm{SU}(2)$ structure on the manifold. Thus, also on Y_6 the possibility of an intermediate $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures does not exist. Instead one can only have an $\mathrm{SU}(2)$ structure or a trivial structure, as we already concluded in our analysis for Y_4 in section 2.3. In the $\mathrm{SU}(2)$ structure case we can write

$$\Phi^+ = \Theta_+ \wedge \Phi_1 \,, \qquad \Phi^- = \Theta_- \wedge \Phi_2 \,, \tag{3.30}$$

where $\Theta_\pm$ are $\mathrm{SO}(2,2)$ spinors of opposite chirality and therefore define a trivial structure on $T_2 Y_6 \oplus T_2^* Y_6$. The $\mathrm{SO}(4,4)$ spinors Φ_1 and Φ_2 are pure and of even chirality and define the $\mathrm{SU}(2)$ structure on $T_4 Y_6 \oplus T_4^* Y_6$. This is precisely the situation we already discussed in section 3.2. There the $\mathrm{SU}(2)$ structure was defined in terms of the two spinors η_i. The relation between the η_i and the $\Phi_\pm$ is analogously to (2.31) described by

$$\Phi^+ = \mathrm{e}^{-B} \wedge \tau(\eta_1 \otimes \bar{\eta}_2) \,, \qquad \Phi^- = \mathrm{e}^{-B} \wedge \tau(\eta_1 \otimes \bar{\eta}_2^c) \,, \tag{3.31}$$

where B is the NS two-form, which is not determined by the η_i. We can insert (2.26) and relate the components in (3.30) to the quantities K, J, B and Ω via (3.8) and (3.9). We end up with

$$\Theta_+ = \mathrm{e}^{-B_{(2)} + \frac{1}{2} K \wedge \bar{K}} \,, \quad \Theta_- = K \,, \quad \Phi_1 = \frac{\mathrm{i}}{4} \mathrm{e}^{-B_{(4)}} \wedge \Omega \,, \quad \Phi_2 = \frac{1}{4} \mathrm{e}^{-B_{(4)} - \mathrm{i} J} \,, \tag{3.32}$$

and therefore

$$\Phi^+ = \frac{\mathrm{i}}{4} \mathrm{e}^{-B_{(2)} + \frac{1}{2} K \wedge \bar{K}} \wedge \mathrm{e}^{-B_{(4)}} \wedge \Omega \,, \qquad \Phi^- = \frac{1}{4} K \wedge \mathrm{e}^{-B_{(4)} - \mathrm{i} J} \,, \tag{3.33}$$

where we denoted the components of the B field on $\Lambda^2 T_2^* Y_6$ by $B_{(2)}$ and on $\Lambda^2 T_4^* Y_6$ by $B_{(4)}$, respectively. As mentioned earlier, there is some gauge freedom in choosing η_1 and η_2 out of the space of SU(2) singlets, which translates into a rotational gauge freedom between Φ_1 and Φ_2. Therefore, it is more convenient in the following not to specify the Φ_i in terms of J and Ω.

3.4 Moduli space of Y_6

Now we are in the position to discuss the moduli space of SU(2) structures on Y_6. By the splitting we described above, we can do this independently for the pure spinors on $T_2 Y_6 \oplus T_2^* Y_6$ and the ones on $T_4 Y_6 \oplus T_4^* Y_6$. On the eight-dimensional subspace $T_4 Y_6 \oplus T_4^* Y_6$ the arguments are the same as on Y_4 and thus Φ_1 and Φ_2 form the moduli space (2.49).

Additionally, the SO(2,2) spinors Θ_+ and Θ_- each parameterize a moduli space on their own. The Lie algebra splits according to

$$so(2,2) = sl(2,\mathbb{R})_T \oplus sl(2,\mathbb{R})_U \,, \tag{3.34}$$

where the generators of the two sub-algebras in (3.34) are given explicitly in (D.6) and (D.7). The first sub-algebra $sl(2,\mathbb{R})_T$ just acts on Θ_+, while the second $sl(2,\mathbb{R})_U$ acts on Θ_-. The degrees of freedom in Θ_+ correspond to a two-form acting on the negative eigenspace of $\mathcal{P}$ and a form of degree zero. Together they form an $Sl(2,\mathbb{R})_T$ doublet which is naturally normalized. Furthermore, we have to mod out the gauge degree of freedom corresponding to the phase of Θ_+. From (2.30) we learn that the remaining complex degree of freedom of Θ_+ is given by the volume and the B-field. It spans the parameter space $Sl(2,\mathbb{R})_T/\mathrm{SO}(2)$. Similarly, Θ_- can be expanded in the basis of one-forms on the negative eigenspace of $\mathcal{P}$, which is two-dimensional, and therefore defines an $Sl(2,\mathbb{R})_U$ doublet analogously to Θ_+, exhibiting the same normalization and gauge degree of freedom. Hence, Θ_- spans the moduli space $Sl(2,\mathbb{R})_U/\mathrm{SO}(2)$.[23]

Additionally to the parameter space of the pure spinors, we have the dilaton field ϕ in the NS-NS sector, which is complexified by the dualized B field in four dimensions, and forms the moduli space $Sl(2,\mathbb{R})_S/\mathrm{SO}(2)$. So altogether we have the (local) moduli space

$$\mathcal{M}_{\Theta_\pm,\Phi_i} = \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)} \times \frac{Sl(2,\mathbb{R})_S}{\mathrm{SO}(2)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \times \frac{Sl(2,\mathbb{R})_U}{\mathrm{SO}(2)} \,. \tag{3.35}$$

To derive the moduli space, we need to perform a Kaluza-Klein truncation as we already discussed in detail in section 2.4. The first factor describes the deformations of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure on $T_4 Y_6 \oplus T_4^* Y_6$, analogously to (2.49). Therefore the first factor leads to the same expression for the global moduli space of the truncated theory as in (2.51). To derive the global moduli space for the other three factors, we note that $T_2 Y_6 \oplus T_2^* Y_6$ and $\Lambda^\bullet T_2^* Y_6$ consist of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ singlets only and thus are trivial bundles. By the general argument on spaces that consist only of structure group singlets which we presented in section 2.4, we conclude that the moduli spaces of $\Theta_\pm$ are unchanged. Since the dilaton

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[23]Note that for $Y_6 = K3 \times T^2, \Theta_\pm$ parameterize the Kähler and complex structure deformations of the T^2, respectively.

and the B-field transform as scalars under the internal Lorentz group of Y_6, their moduli space also stays the same after Kaluza-Klein truncation. Therefore, the moduli space of the Kaluza-Klein truncated theory is

$$\mathcal{M}^{\text{NS}}_{d=4} = \frac{\text{SO}(4, n+4)}{\text{SO}(4) \times \text{SO}(n+4)} \times \frac{Sl(2, \mathbb{R})_S}{\text{SO}(2)} \times \frac{Sl(2, \mathbb{R})_T}{\text{SO}(2)} \times \frac{Sl(2, \mathbb{R})_U}{\text{SO}(2)} . \tag{3.36}$$

So far we left $\mathcal{P}$ fixed for the whole discussion. Indeed, in appendix C we show that projecting out all massive gravitinos also eliminates all degrees of freedom that correspond to deformations of $\mathcal{P}$. Therefore, (3.36) indeed describes the NS moduli space of SU(2) structure compactifications in $d = 4$.

As we already noted at the end of section 2.4, the components of the exterior derivatives $\mathrm{d}\Phi_{1,2}$ which are in $\Lambda^{\text{odd}} T_4^* Y_6$, i.e. with all legs in the directions of the four-dimensional part of the tangent space, are SU(2) doublets and therefore in the absence of a warp factor are projected out. This additionally constrains Y_6 in that its four-dimensional component has to be a $K3$ manifold. Or in other words Y_6 has to be a $K3$ fibered over a (twisted) torus.[24] This situation was already analyzed in ref. [23] and we will return to it in [27].

3.5 R-R scalars and exceptional generalized geometry

Now we want to include the RR-fields by extending the pure spinor formalism again to exceptional generalized geometry. We will mainly use the Ansatz proposed in [21] but applied to the situation of some SU(2)-structure. The main difference will be the existence of a generalized almost product structure $\mathcal{P}$ which has already been introduced above and the projection procedure to $N = 4$.

The U-duality group in $d = 4$ is $E_{7(7)}$ with the T-duality subgroup being $\text{SO}(6,6)$. Let us first recall the decomposition of the representations of $E_{7(7)}$ in terms of the maximal subgroup $Sl(2, \mathbb{R})_S \times \text{SO}(6,6)$. The factor $Sl(2, \mathbb{R})_S$ is the S-duality subgroup acting on the four-dimensional dilaton ϕ complexified by the dualized B-field. The fundamental representation of $E_{7(7)}$ decomposes as [45]

$$\mathbf{56} \rightarrow (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) , \tag{3.37}$$

while the adjoint of $E_{7(7)}$ decomposes as

$$\mathbf{133} \rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{66}) + (\mathbf{2}, \bar{\mathbf{32}}) . \tag{3.38}$$

As in section 2.5, we can derive the geometrical realization of these representations by considering the charges in $d = 4$. It was shown in ref. [43] that the electric and magnetic charges form the $\mathbf{56}$ representation of $E_{7(7)}$. The $(\mathbf{2}, \mathbf{12})$ part in (3.37) represents the NS-NS charges, i.e. winding and momentum modes as well as NS5-branes and KK-monopoles, and thus corresponds to a doublet in $TY_6 \oplus T^* Y_6$.[25] The $(\mathbf{1}, \mathbf{32})$ represents the RR charges,

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[24]We thank Danny Martínez-Pedrera for discussions on this issue.

[25]In contrast to [18], we do not distinguish the bundles $TY_6 \oplus T^* Y_6$ and $\Lambda^5 TY_6 \oplus \Lambda^5 T^* Y_6$ because they are related by a volume form on Y_6. Such a volume form we already chose to identify the SO(6,6) spinor bundles with $\Lambda^{\text{even}} T^* Y_6$ and $\Lambda^{\text{odd}} T^* Y_6$. Thus, we can identify the bundles $TY_6 \oplus T^* Y_6$ and $\Lambda^5 TY_6 \oplus \Lambda^5 T^* Y_6$, and write them as a doublet under the S-duality group.

which correspond to ten-dimensional D-brane solutions. In type IIA, they are elements of $\Lambda^{\text{even}}T^*Y_6$, while in type IIB they live in the bundle $\Lambda^{\text{odd}}T^*Y_6$ [43]. Therefore, (3.37) is realized geometrically by [18]

$$\mathbf{56}^{\text{IIA}} = (TY_6 \oplus T^*Y_6)_{\mathbf{2}} \oplus (\Lambda^{\text{even}}T^*Y_6)_{\mathbf{1}} \tag{3.39}$$

for type IIA, and by

$$\mathbf{56}^{\text{IIB}} = (TY_6 \oplus T^*Y_6)_{\mathbf{2}} \oplus (\Lambda^{\text{odd}}T^*Y_6)_{\mathbf{1}} \tag{3.40}$$

in type IIB. The subscript denotes the representation under the S-duality group $Sl(2,\mathbb{R})_S$, which has no geometric realization. Correspondingly, the decomposition of the adjoint of the U-duality group is realized geometrically by

$$\mathbf{133}^{\text{IIA}} = (\mathbb{R})_{\mathbf{3}} \oplus (so(TY_6 \oplus T^*Y_6))_{\mathbf{1}} \oplus (\Lambda^{\text{odd}}T^*Y_6)_{\mathbf{2}} \tag{3.41}$$

for type IIA, and

$$\mathbf{133}^{\text{IIB}} = (\mathbb{R})_{\mathbf{3}} \oplus (so(TY_6 \oplus T^*Y_6))_{\mathbf{1}} \oplus (\Lambda^{\text{even}}T^*Y_6)_{\mathbf{2}} \tag{3.42}$$

for type IIB. As on four-dimensional manifolds Y_4, we see that the spinor representations of $SO(6,6)$ are related to the RR-fields C. In type IIA, the C-fields define an $SO(6,6)$ spinor of odd chirality via[26]

$$C^{IIA} = C_1 + C_3 + C_5 \ \in \Lambda^{\text{odd}}T^*Y_6\,, \tag{3.43}$$

while in type IIB, the spinor is of even chirality and defined by

$$C^{IIB} = C_0 + C_2 + C_4 + C_6 \ \in \Lambda^{\text{even}}T^*Y_6\,. \tag{3.44}$$

This fits nicely with the $SO(6,6)$ spinors appearing in (3.41) and (3.42). However, in both (3.41) and (3.42) there appears a doublet of $SO(6,6)$ spinors in the adjoint of $E_{7(7)}$. As we will see below, one linear combination of these spinors is in the stabilizer of the $SU(2) \times SU(2)$ structure, while the remaining linearly independent linear combination corresponds to the RR scalar fields.

We have argued in section 3.1 that in order to compactify on a non-trivial $SU(2)\times SU(2)$ structure, we have to project out all $SU(2) \times SU(2)$ doublets. We already know from section 3.3 that only the $SO(2,2) \times SO(4,4)$ subgroup of $SO(6,6)$ survives this projection. Therefore, the $(\mathbf{1},\mathbf{66})$ component in (3.38) is projected to the direct sum of the adjoints of $SO(2,2)$ and $SO(4,4)$. Furthermore, the first component in (3.38) consists of $SO(6,6)$ singlets. Therefore, it is also a singlet under $SU(2) \times SU(2)$ and thus invariant under the projection. Hence, we are left with the last component in (3.38), which is a doublet of $SO(6,6)$ spinors.

Due to the existence of the generalized almost product structure $\mathcal{P}$, we can decompose the $SO(6,6)$ spinor bundles as done in (3.29). Furthermore, as shown in section 2.4 we

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[26]Note that we use the "democratic" formulation for the RR-fields, and that we only consider scalar degrees of freedom. Therefore, all legs of the forms in (3.43) and (3.44) are internal.

know that the projection to $N = 4$ removes $\Lambda^{\mathrm{odd}}_4 T^* Y_6$ and we are left with only half of the degrees of freedom

$$
\begin{aligned}
\Lambda^{\mathrm{even}} T^* Y_6 &\longrightarrow \Lambda^{\mathrm{even}} T^*_2 Y_6 \wedge \Lambda^{\mathrm{even}} T^*_4 Y_6 \,, \\
\Lambda^{\mathrm{odd}} T^* Y_6 &\longrightarrow \Lambda^{\mathrm{odd}} T^*_2 Y_6 \wedge \Lambda^{\mathrm{even}} T^*_4 Y_6 \,.
\end{aligned}
\tag{3.45}
$$

Therefore, only part of the U-duality group $E_{7(7)}$ survives this projection. In type IIA, we end up with the subgroup G^{IIA} whose adjoint is the subspace of (3.41) given by

$$
g^{\mathrm{IIA}} = (\mathbb{R})_{\mathbf{3}} \oplus so(T_2 Y_6 \oplus T^*_2 Y_6)_{\mathbf{1}} \oplus so(T_4 Y_6 \oplus T^*_4 Y_6)_{\mathbf{1}} \oplus (\Lambda^{\mathrm{odd}} T^*_2 Y_6 \wedge \Lambda^{\mathrm{even}} T^*_4 Y_6)_{\mathbf{2}} \,. \tag{3.46}
$$

In type IIB, we find the subgroup G^{IIB} whose adjoint is the subspace of (3.42) given by

$$
g^{\mathrm{IIB}} = (\mathbb{R})_{\mathbf{3}} \oplus so(T_2 Y_6 \oplus T^*_2 Y_6)_{\mathbf{1}} \oplus so(T_4 Y_6 \oplus T^*_4 Y_6)_{\mathbf{1}} \oplus (\Lambda^{\mathrm{even}} T^*_2 Y_6 \wedge \Lambda^{\mathrm{even}} T^*_4 Y_6)_{\mathbf{2}} \,. \tag{3.47}
$$

In appendix D.1 we show that both G^{IIA} and G^{IIB} define $\mathrm{SO}(6,6) \times Sl(2,\mathbb{R})_{T/U}$ subgroups of $E_{7(7)}$. The $Sl(2,\mathbb{R})_{T/U}$ factor is generated by one of the two sub-algebras in (3.34), depending on whether one considers type IIA or type IIB. $Sl(2,\mathbb{R})_T$ acts on the Kähler part of the identity structure on $T_2 Y_6 \oplus T^*_2 Y_6$ and forms the extra factor in type IIA, while $Sl(2,\mathbb{R})_U$ acts on its complex structure part and forms the extra factor in type IIB.

3.6 Including the R-R sector

Finally, in this section we determine the moduli space of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure compactifications of type II theories in $d = 4$. We will mainly consider type IIA but the type IIB results are easily obtained by changing some chiralities and an exchange of $sl(2,\mathbb{R})_T$ with $sl(2,\mathbb{R})_U$ in the Lie algebra $so(2,2)$. (For more details, see appendix D.1.)

We start by embedding the pure $\mathrm{SO}(6,6)$ spinors into some $E_{7(7)}$ representations. The spinor Φ^+ of positive chirality is embedded into the fundamental representation via

$$
\lambda = (\lambda^A_i, \lambda^+) = (0, \mathrm{Re}(\Phi^+)) \,, \quad i = 1, 2 \,, \tag{3.48}
$$

where we used the decomposition (3.37) and (D.1).[27] The stabilizer of λ is determined in appendix D.2.1 to be $\mathrm{SO}(4,6) \times \mathrm{SO}(2) \subset \mathrm{SO}(6,6) \times Sl(2,\mathbb{R})_T$. Furthermore the phase of Φ^+ is just some gauge degree of freedom. In the $E_{7(7)}$ covariant formalism this gauge freedom manifests itself in the fact that λ and

$$
\tilde{\lambda} = (0, \mathrm{Im}(\Phi^+)) \tag{3.49}
$$

describe the same $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure. They are related by the generalized almost complex structure $\mathcal{J}_{\Phi^+}$ which embeds into the adjoint of $E_{7(7)}$. Therefore, after modding out the transformations generated by $\mathcal{J}_{\Phi^+}$, the parameter space for λ is

$$
\mathcal{M}_\lambda = \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(2) \times \mathrm{SO}(4,6)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \,. \tag{3.50}
$$

In appendix D.2.1 we determine the action of the $E_{7(7)}$ transformations on the embedding (3.48).

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[27]Note that pure $\mathrm{SO}(6,6)$ spinors are completely determined by their real or imaginary part (or any linear combination) [6].

The pure $SO(6,6)$ spinor of negative chirality cannot be embedded into the fundamental of $E_{7(7)}$ but only into its adjoint. However, we see from (3.38) that we must embed it as an $Sl(2,\mathbb{R})_S$ doublet. Therefore, we introduce some complex vector $u^i, i = 1, 2$, which is stable and normalized, i.e.

$$|u|^2 = u^i \epsilon_{ij} \bar{u}^j = 1 . \tag{3.51}$$

u^i describes the complexified dilaton degree of freedom. Then we embed Φ^- via

$$\mu_1 = (\hat{\mu}^i{}_j, \mu^A{}_B, \mu^{i-}) = (0, 0, \text{Re}(u^i \Phi^-)) . \tag{3.52}$$

Note that the overall phase of u^i is just a choice of gauge. The calculation of the moduli space however is a bit more involved than expected. Naively one would think that analogously to the gauge freedom of λ the gauge freedom in μ_1 is some phase rotation which relates μ_1 to

$$\mu_2 = (0, 0, \text{Im}(u^i \Phi^-)) . \tag{3.53}$$

However, these two elements of the adjoint do not commute, and therefore determine a third one which reads

$$\mu_3 = \frac{\mathrm{i}}{4k} \langle \bar{\Phi}^-, \Phi^- \rangle (u^i \bar{u}_j + \bar{u}^i u_j, \mathrm{i}\,|u|^2 (\mathcal{J}_{\Phi^-})^A{}_B, 0) , \tag{3.54}$$

where $\mathcal{J}_{\Phi^-}$ is the generalized almost complex structure corresponding to Φ^- and defined by [13]

$$(\mathcal{J}_{\Phi^-})_{AB} = \mathrm{i} \frac{\langle \bar{\Phi}^-, \Gamma_{AB} \Phi^- \rangle}{\langle \bar{\Phi}^-, \Phi^- \rangle} . \tag{3.55}$$

The normalization k is defined as

$$k = \sqrt{\frac{1}{2} |u|^2 \langle \bar{\Phi}^-, \Phi^- \rangle} . \tag{3.56}$$

As explained in section 2.3.1, $\mathcal{J}_{\Phi^-}$ determines Φ^- up to a phase. As a consequence, μ_3 determines μ_1 and μ_2 up to a rotation between those two. Hence, each μ_a determines the other two. It turns out that the μ_a define a highest weight $SU(2)$ embedding of u^i and the pure spinor Φ^- in $E_{7(7)}$ [21, 30]. Indeed, the μ_a fulfill the $su(2)$ algebra

$$[\mu_a, \mu_b] = 2k \epsilon_{abc} \mu_c . \tag{3.57}$$

Purity of Φ^- together with (3.51) is equivalent to the fact that the μ_a indeed form an $su(2)$ algebra. Furthermore, the μ_a share the same stabilizer and make the $SU(2)$ gauge freedom manifest. By calculating the stabilizer and modding out all gauge degrees of freedom, we end up with the parameter space

$$\mathcal{M}_\mu = \frac{SO(6,6)}{SO(4) \times SO(2,6)} . \tag{3.58}$$

The calculations are a bit lengthy and are presented in appendix D.2.2. There, we also give the action of the additional $E_{7(7)}$ transformations on the μ_a.

Finally, we consider the common parameter space of both objects. By imposing [21]

$$\mu_a \cdot \lambda = 0 \,, \tag{3.59}$$

we generalize the usual compatibility conditions (3.17). We compute in appendix D.2.3 that the (local) moduli space is

$$\mathcal{M}_{\lambda,\mu} = \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6) \times \mathrm{SO}(6)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \,, \tag{3.60}$$

where we also modded out the additional gauge transformations that arise as symmetries between the $\Lambda_4^{\mathrm{even}} T^* Y$ components of Φ^+ and Φ^-, as we already did in the case of compactification to six dimensions. The embeddings (3.48) and (3.52) can be deformed by some $E_{7(7)}$ transformation. This corresponds to the $\mathrm{SO}(6,6)$ deformations which we discussed already in the last sections and to additional degrees of freedom that can be identified with the RR scalars.

It is now easy to understand the moduli space. For this we note that the subspace of positive signature in the first factor in (3.60) is spanned by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ singlets. By applying the general argument presented in section 2.4, we know that after the Kaluza-Klein truncation this subspace is still of dimension six. However, the space of negative signature is spanned by $(\mathbf{2}, \mathbf{2})$ representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Therefore, its dimension can be different globally, say $n+6$. For the second factor, we can argue in the same way as in section 3.4 to show that it gives the same moduli space globally. Therefore, the moduli space is

$$\mathcal{M}_{d=4}^{IIA} = \frac{\mathrm{SO}(6, 6+n)}{\mathrm{SO}(6) \times \mathrm{SO}(6+n)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \,. \tag{3.61}$$

In appendix D.1 we give a prescription how to obtain the corresponding type IIB moduli space

$$\mathcal{M}_{d=4}^{IIB} = \frac{\mathrm{SO}(6, 6+n)}{\mathrm{SO}(6) \times \mathrm{SO}(6+n)} \times \frac{Sl(2,\mathbb{R})_U}{\mathrm{SO}(2)} \,. \tag{3.62}$$

Similar to the interpretation of the pure spinor formalism as hyperkähler $\mathrm{SU}(2)$ quotient in the case of $d = 6$, which we discussed in appendix B.1, we can interpret the purity and compatibility conditions as leading from the superconformal cone to the moduli space of supergravity. This is discussed in more detail in appendix B.2.

4 Conclusions

In this paper we showed that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures always reduce to either $\mathrm{SU}(2)$ or identity structures and we derived the general form of the moduli space for all $\mathrm{SU}(2)$ backgrounds in four, five and six space-time dimensions.

For compactifications to six dimensions, we derived the geometric subspace of scalar degrees of freedom to be

$$\mathcal{M}_{d=6}^{\mathrm{geom}} = \frac{\mathrm{SO}(3, n+3)}{\mathrm{SO}(3) \times \mathrm{SO}(n+3)} \times \mathbb{R}_+ \,, \tag{4.1}$$

where n is some integer number and the $\mathbb{R}_+$ corresponds to the volume factor. We included the B field into this moduli space by switching to the framework of generalized geometry. The dilaton gives an extra degree of freedom that does not mix with the rest of the NS-sector and we found that the NS moduli space reads

$$\mathcal{M}_{d=6}^{\mathrm{NS}} \;=\; \frac{\mathrm{SO}(4, n+4)}{\mathrm{SO}(4) \times \mathrm{SO}(n+4)} \times \mathbb{R}_+ \; . \tag{4.2}$$

We also argued in appendix B.1 that the pure spinor formalism describes the analogue of a hyperkähler SU(2) quotient in $d = 6$ and that the corresponding hyperkähler potential can be naturally expressed in terms of pure spinors.

We further showed that by embedding both pure spinors into representations of the U-duality group $\mathrm{SO}(5,5)$ we are able to incorporate the RR-scalars into the moduli space. We showed that in the IIA case no RR-scalars arise and the moduli space is still given by (4.2), i.e. $\mathcal{M}_{d=6}^{\mathrm{IIA}} = \mathcal{M}_{d=6}^{\mathrm{NS}}$. For type IIB additional RR-scalars exist and they enlarge the moduli space to

$$\mathcal{M}_{d=6}^{\mathrm{IIB}} \;=\; \frac{\mathrm{SO}(5, n+5)}{\mathrm{SO}(5) \times \mathrm{SO}(n+5)} \; . \tag{4.3}$$

We used the same strategy to determine the moduli spaces for SU(2) structure compactifications to $d = 4$. Additionally, we had to introduce an almost product structure and generalizations thereof to force the structure group to be SU(2). We then derived the geometric moduli space of these compactifications to be

$$\mathcal{M}_{d=4}^{\mathrm{geom}} = \frac{\mathrm{SO}(3, 3+n)}{\mathrm{SO}(3) \times \mathrm{SO}(3+n)} \times \mathbb{R}_+ \times \frac{Sl(2,\mathbb{R})}{\mathrm{SO}(2)} \times \mathbb{R}_+ \; . \tag{4.4}$$

Incorporating the B fields of the NS-sector enlarges the moduli space to

$$\mathcal{M}_{d=4}^{\mathrm{NS}} = \frac{\mathrm{SO}(4, n+4)}{\mathrm{SO}(4) \times \mathrm{SO}(n+4)} \times \frac{Sl(2,\mathbb{R})_S}{\mathrm{SO}(2)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \times \frac{Sl(2,\mathbb{R})_U}{\mathrm{SO}(2)} \; . \tag{4.5}$$

Within the $E_{7(7)}$ covariant formalism of exceptional generalized geometry we derived the moduli space to be of the form

$$\mathcal{M}_{d=4}^{\mathrm{IIA/IIB}} = \frac{\mathrm{SO}(6, n+6)}{\mathrm{SO}(6) \times \mathrm{SO}(n+6)} \times \frac{Sl(2,\mathbb{R})_{T/U}}{\mathrm{SO}(2)} \; , \tag{4.6}$$

where the extra factor is either $\frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)}$ or $\frac{Sl(2,\mathbb{R})_U}{\mathrm{SO}(2)}$ depending on whether we consider type IIA or type IIB. We also showed that we can interpret the flat superconformal cone over this space in terms of pure spinors and their embeddings into $E_{7(7)}$ representations. Furthermore, we explicitly identified how shifts in the fields correspond to shifts in the moduli space and determined how the RR-scalars are fibered over the rest of the space. We also briefly commented on the appearance of string/string/string triality [46] in these compactifications.

In the appendix we performed the analogous analysis for compactifications to $d = 5$. We argued that the moduli space is the same for type IIA and IIB due to the absence of chirality in five dimensions and — after incorporating all scalar degrees of freedom — reads

$$\mathcal{M}_{d=5} = \frac{\mathrm{SO}(5, n+5)}{\mathrm{SO}(5) \times \mathrm{SO}(n+5)} \times \mathbb{R}_+ \; . \tag{4.7}$$

Note that this work covers all compactifications of type II theories that lead to 16 supercharges in the low energy effective action. The moduli spaces which we derived here could already have been predicted from the general form of supergravity theories with 16 supercharges. However, here we showed explicitly how these moduli spaces arise in the compactification procedure. More precisely, we gave an example how the U-duality covariant formalism can be used to determine the moduli space for backgrounds that break part of the supersymmetry.

In the derivation we mainly used algebraic properties of the pure spinors but did not impose explicitly any differential constraint. The reason being that the metric on scalar field space is determined by the algebraic properties while differential constraints affect the potential of the effective action. However, by analyzing the light spectrum of the effective supergravity we argued that we have to project out all SU(2) doublet degrees of freedom in order to remove the massive gravitino multiplets. Their presence would alter the standard supergravity with 16 supercharges and in particular change the scalar geometry. Since the exterior derivative of the pure spinors $d\Phi$ is an SU(2) doublet this effectively also constrains the class of compactification manifolds. In the absence of a warp factor it implies that $K3$ is the four-dimensional compactification manifold Y_4, while for the higher-dimensional $Y_{5,6}$ a component of the almost product structure appears locally as $K3$.

For all spaces the number of light modes is determined by the integer n with $n = 16$ for $K3$. Generically, this number is related to the global twisting of the bundle of forms that are in the $(\mathbf{2}, \mathbf{2})$ representation of $SU(2) \times SU(2)$. All other details of the dimensional reduction are encoded in the possible gauging of the supergravity action and in the warp factor. It would be interesting to analyze the possible differential conditions induced by consistent flux configurations and to classify the gauged supergravities that can arise as low energy effective actions. A first step in this direction will be presented in [27]. It would also be interesting to study possible non-perturbative dualities with the heterotic string as was done, for example, in ref. [23].

Acknowledgments

A Spinor conventions and Fierz identities

In this appendix we collect our spinor conventions and Fierz identities used throughout the paper. For $SO(N)$ the gamma-matrices γ_m satisfy

$$\{\gamma_m, \gamma_n\} = 2g_{mn}, \quad m, n = 1, \ldots, N, \tag{A.1}$$

where g_{mn} is the $SO(N)$ metric, which can be used to raise and lower the index of the gamma-matrices. For N even the chirality operator is given by $\gamma_0 =$

$\mathrm{i}^{N/2} \frac{1}{N!} \epsilon^{m_1 \dots m_N} \gamma_{m_1 \dots m_N}$, where ϵ specifies the orientation of the manifold. For antisymmetric products of gamma-matrices we abbreviate

$$\gamma_{m_1 \dots m_k} = \gamma_{[m_1} \cdots \gamma_{m_k]} \,. \tag{A.2}$$

The antisymmetric products of two gamma-matrices γ_{mn} fulfill the $SO(N)$ commutation relations and generate the action of $SO(N)$ on spinors η.

As explained e.g. in [47], for any N one can define the charge conjugation matrix, which maps a spinor η to its charge conjugate η^c. For $N = 4k, k \in \mathbb{N}_0$, the charge conjugation matrix commutes with the chirality operator and therefore charge conjugation preserves the chirality of a spinor. For $N = 4k + 2$, charge conjugation anti-commutes with the chirality operator and thus exchanges the chirality of spinors.

With (2.9) and (2.25) we can compute the $SO(4)$ Fierz identities to be

$$\begin{aligned}
\eta_\alpha \bar\eta^\beta &= \frac{1}{2}(P_-)_\alpha{}^\beta - \frac{1}{8}\mathrm{i}\, J_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \,, \qquad \text{for } \alpha, \beta = 1, \dots, 4 \,, \\
(\eta^c)_\alpha \bar\eta^\beta &= \frac{1}{8}\mathrm{i}\, \bar\Omega_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \,, \\
\eta_\alpha (\bar\eta^c)^\beta &= \frac{1}{8}\mathrm{i}\, \Omega_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \,,
\end{aligned} \tag{A.3}$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$ are the chiral projection operators.

Analogously, the $SO(6)$ Fierz identities for two spinors η_1 and η_2 can be derived by using (2.25) together with the definitions (3.6), (3.8) and (3.9) to be

$$\begin{aligned}
(\eta_i)_\alpha (\bar\eta_i)^\beta &= \frac{1}{2}(P_-)_\alpha{}^\beta - \frac{1}{4}\mathrm{i}\, J^{(i)}_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \qquad \text{for } i = 1, 2, \text{ and } \alpha, \beta = 1, \dots, 8 \,, \\
(\eta_i^c)_\alpha (\bar\eta_i)^\beta &= \frac{1}{24}\mathrm{i}\, \bar\Omega^{(i)}_{mnp}\,(\gamma^{mnp}P_-)_\alpha{}^\beta \,, \\
(\eta_i)_\alpha (\bar\eta_i^c)^\beta &= \frac{1}{24}\mathrm{i}\, \Omega^{(i)}_{mnp}\,(\gamma^{mnp}P_+)_\alpha{}^\beta \,, \\
(\eta_1)_\alpha (\bar\eta_2^c)^\beta &= \frac{1}{2}K_m\,(\gamma^m P_+)_\alpha{}^\beta - \frac{1}{8}\mathrm{i}\, K_m J_{np}\,(\gamma^{mnp}P_+)_\alpha{}^\beta \,, \\
(\eta_2^c)_\alpha (\bar\eta_1)^\beta &= \frac{1}{2}\bar K_m\,(\gamma^m P_-)_\alpha{}^\beta - \frac{1}{8}\mathrm{i}\, \bar K_m J_{np}\,(\gamma^{mnp}P_-)_\alpha{}^\beta \,, \\
(\eta_2)_\alpha (\bar\eta_1)^\beta &= \frac{1}{4}\mathrm{i}\, \bar\Omega_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \,, \\
(\eta_1)_\alpha (\bar\eta_2)^\beta &= \frac{1}{4}\mathrm{i}\, \Omega_{mn}\,(\gamma^{mn}P_-)_\alpha{}^\beta \,.
\end{aligned} \tag{A.4}$$

With the help of

$$(\eta_1)_\alpha (\bar\eta_1)^\beta = (\eta_1)_\alpha (\bar\eta_2^c)^\delta (\eta_2^c)_\delta (\bar\eta_1)^\beta \,, \qquad (\eta_1)_\alpha (\bar\eta_1^c)^\beta = (\eta_1)_\alpha (\bar\eta_2)^\delta (\eta_2)_\delta (\bar\eta_1^c)^\beta \,, \tag{A.5}$$

etc., we can derive the relations (3.10) and (3.7) for the forms involved.

For $SO(N, N)$ spinors, the gamma-matrices Γ_A are defined by

$$\{\Gamma_A, \Gamma_B\} = 2\mathcal{I}_{AB} \,, \quad A, B = 1, \dots, 2N \,, \tag{A.6}$$

where $\mathcal{I}$ is the $SO(N,N)$ metric. We can also write the gamma-matrices in terms of raising and lowering operators Γ_{m+} and Γ_{m-} such that

$$\begin{aligned}
\{\Gamma_{m+}, \Gamma_{n+}\} &= 0\,, \\
\{\Gamma_{m-}, \Gamma_{n-}\} &= 0\,, \\
\{\Gamma_{m+}, \Gamma_{n-}\} &= 2g_{mn} \quad \text{for all } m,n = 1,\ldots,N\,,
\end{aligned} \tag{A.7}$$

where g_{mn} is the $SO(N)$ metric. As for $SO(N)$ gamma-matrices, we abbreviate the anti-symmetric product of $SO(N,N)$ gamma-matrices by

$$\Gamma_{A_1\ldots A_k} = \Gamma_{[A_1}\ldots\Gamma_{A_k]}\,. \tag{A.8}$$

The antisymmetric products of two gamma-matrices Γ_{AB} fulfill the $SO(N,N)$ commutation relations, and generate the action of $SO(N,N)$ on spinors Φ. The chirality operator is given by $\Gamma_0 = \frac{1}{(2N)!}\epsilon^{A_1\ldots A_d}\Gamma_{A_1\ldots A_d}$, where ϵ is naturally normalized by

$$\epsilon^{m_1+m_1-\ldots m_N+m_N-} = 1\,, \tag{A.9}$$

if N is even. In this case it defines a canonical choice of positive chirality.

Over a point on a k-dimensional manifold Y_k we can define $SO(k,k)$ gamma-matrices via the operators

$$\Gamma_{m+} \equiv \mathrm{d}x^m\wedge\,, \quad \Gamma_{m-} \equiv \iota_{x^m}\,, \tag{A.10}$$

which act on forms and where ι_{x_m} denotes the insertion of the tangent vector x_m. They naturally fulfill the Clifford algebra (A.7) since

$$[\mathrm{d}x^m\wedge,\ \iota_{x^n}]\omega_p = \delta^m{}_n\omega_p\,, \tag{A.11}$$

for any p-form ω_p. Therefore, we can canonically define an $SO(k,k)$ action on the space of forms $\Lambda^\bullet T^*Y_k$. The chirality operator Γ_0 acts on a p-form ω_p by

$$\Gamma_0\,\omega_p = (-1)^p\,\omega_p\,, \tag{A.12}$$

hence the Weyl spinor bundle of positive (negative) chirality is given by the bundle of even (odd) forms. The generators of this $SO(k,k)$ action naturally split into three types according to the number of raising and lowering operators. Transformations of the type Γ_{m+n-} preserve the degree of a form and span the algebra of the geometrical group $Gl(k,\mathbb{R})$. The generators Γ_{m+n+} and Γ_{m-n-} correspond to two-forms and bi-vectors. Hence we conclude

$$so(k,k) = gl(k,\mathbb{R}) \oplus \Lambda^2 T^*Y \oplus \Lambda^2 TY\,. \tag{A.13}$$

B Superconformal cones

In this appendix we briefly discuss the existence of various higher-dimensional manifolds that arise as cones over the moduli spaces which we discussed in the main text. These cones appear in the associated superconformal supergravity where the extra degrees of freedom correspond to a superconformal compensating multiplet.

In theories with eight supercharges ($N = 2$ in $d = 4$) the scalars of the hypermultiplets span a quaternionic manifold which is the base of a hyperkähler cone (the Swann bundle) [30]. The construction of the cone is well understood in $N = 2$ superconformal supergravity [31].

The NS-sector of type II compactification is shared by a compactification of the heterotic string on the same manifold, which therefore only has half the amount of supersymmetry. As a consequence the moduli spaces $\mathcal{M}^{\mathrm{NS}}_{d=6,5,4}$ given in (2.51), (E.17), (3.36) all contain the same manifold $\mathrm{SO}(4, n + 4)/\mathrm{SO}(4) \times \mathrm{SO}(4 + n)$. The associated hyperkähler cone of the $N = 2$ superconformal supergravity is briefly reviewed in appendix B.1.

The superconformal supergravity for sixteen supercharges ($N = 4$ in $d = 4$) is less developed [26]. However, on general grounds one expects the existence of a flat cone over the $N = 4$ scalar manifold. This flat cone is briefly discussed in appendix B.2.

B.1 The hyperkähler cone

Let us consider the moduli space of $\mathrm{SU}(2)$ structures on four-dimensional manifolds Y_4, which we argued in section 2.4 to look like (2.51). Here, we want to discuss how specific geometrical constructions over this moduli space arise naturally in the formalism of pure spinors.

Let us start with four real unconstrained but linearly independent spinors Ψ_a, $a = 1, \ldots, 4$, living in the space $\Lambda^{\mathrm{even}}_{\mathrm{finite}} T^* Y_4$ of signature $(4, n + 4)$. These Ψ_a can be associated with the spinors we defined above (2.48) but without imposing any purity and compatibility conditions on them. They parameterize an open subset in the space $\bigoplus_{i=1}^{4} \mathbb{R}^{4,n+4}$, which is just a flat cone over (2.51). The purity and compatibility conditions (2.48) then lead to the deformation space

$$\frac{\mathrm{SO}(4, n + 4)}{\mathrm{SO}(n + 4)} \times \mathbb{R}_+ \,, \tag{B.1}$$

where the $\mathbb{R}_+$ factor corresponds to the gauge freedom to choose the parameter c.

In order to construct the physical moduli space, we mod out an additional $\mathrm{SO}(4)$ between the spinors and the overall $\mathbb{R}_+$ factor. The generators $(\mathcal{J}_{ij})^A{}_B$ for this $\mathrm{SO}(4)$ are given by

$$\mathcal{J}^{AB}_{ij} = \langle \Psi_i, \Gamma^{AB} \Psi_j \rangle \,, \quad (\mathcal{J}_{ij})^A{}_B = \mathcal{J}^{AC}_{ij} \mathcal{I}_{CB} \,. \tag{B.2}$$

They span the Lie algebra $su_1(2) \oplus su_2(2)$, where

$$su_1(2) = \mathrm{span}(\mathcal{J}_{ij} + \epsilon_{ijkl} \mathcal{J}_{kl}) \,, \tag{B.3}$$

$$su_2(2) = \mathrm{span}(\mathcal{J}_{ij} - \epsilon_{ijkl} \mathcal{J}_{kl}) \,. \tag{B.4}$$

If we mod out only the $\mathrm{SU}(2)$ group generated by $su_2(2)$, we end up with the non-Abelian hyperkähler $\mathrm{SU}(2)$ quotient of flat space [41]. This is the hyperkähler cone

$$\frac{\mathrm{SO}(4, n + 4)}{\mathrm{SU}(2) \times \mathrm{SO}(n + 4)} \times \mathbb{R}_+ \tag{B.5}$$

of the quaternionic manifold given by (2.51) [30, 41, 42]. By comparing our calculations with those in [42], one easily sees that (2.40), (2.38) and (2.41) are just the conditions to

set all nine moment maps of the SU(2) generated by $su_2(2)$ to zero. The three generators of $su_1(2)$ define the hyperkähler structure of (B.5). The hyperkähler potential of the space $\bigoplus_{i=1}^{4} \mathbb{R}^{4,n+4}$ we started with is simply

$$H = \langle \bar{\Phi}_1, \Phi_1 \rangle + \langle \bar{\Phi}_2, \Phi_2 \rangle = J \wedge J + \Omega \wedge \bar{\Omega} . \tag{B.6}$$

By performing the hyperkähler SU(2) quotient, one derives the hyperkähler potential of (B.5), which is [42]

$$H = 2\sqrt{\langle \bar{\Phi}_1, \Phi_1 \rangle \langle \bar{\Phi}_2, \Phi_2 \rangle - \langle \bar{\Phi}_1, \Phi_2 \rangle \langle \bar{\Phi}_2, \Phi_1 \rangle} . \tag{B.7}$$

Here, the second term in the square root is included to make the expression SO(4) invariant. This hyperkähler potential also determines the metric on the quaternionic manifold (2.51).

B.2 Flat cone of $N = 4$

For (non-chiral) theories with 16 supercharges in d space-time dimensions the moduli space is given by $\mathcal{M} = \frac{\text{SO}(10-d,n_V)}{\text{SO}(10-d) \times \text{SO}(n_V)} \times \mathbb{R}^+$, where n_V counts the number of vector multiplets [26]. The gravitational multiplet contains $(10-d)$ vectors while each vector multiplet contains one vector and $(10-d)$ scalars. $\mathcal{M}$ is thus spanned by the $n_V \cdot (10-d)$ scalars of the vector multiplets.

In the superconformal framework one needs to couple $(10-d)$ additional compensating vector multiplets with $(10-d)^2$ additional scalars. Thus, the scalar manifold in the superconformal supergravity has dimension $(n_V + (10-d)) \cdot (10-d)$. Since theories with 16 supercharges only have the gauge couplings as free parameters, one expects the superconformal theory to have a flat scalar manifold. Or in other words, one expects the existence of a flat $(n_V + (10-d)) \cdot (10-d)$-dimensional cone over $\mathcal{M}$.[28] Indeed $\mathcal{M}$ can be viewed as the choice of a space-like $(10-d)$-dimensional subspace inside a space of signature $(10-d, n_V)$. If the positions of the vectors which span this subspace are unconstrained they define a $Gl(10-d)$-bundle over $\mathcal{M}$, which is an open subset in $\bigoplus_{i=1}^{(10-d)} \mathbb{R}^{(10-d,n_v)}$.

In the construction of the moduli spaces (2.51), (E.53) and (3.61) we deal each time with a set of $(10-d)$ space-like vectors in a $(10-d, n_V)$-dimensional space as soon as we project to a theory with sixteen supercharges. These space-like vectors span in each case the superconformal cone over the moduli space. By imposing purity and compatibility conditions for these vectors and modding out the rotational symmetry between them, we project this cone to the underlying moduli space of the corresponding supergravity theory. Let us discuss this now for each case in more detail.

In appendix B.1 we already explained how the four real vectors Ψ_a defined above (2.48) parameterize the flat cone over the first factor of (2.51) and how the purity and compatibility conditions together with the removal of gauge freedom project this cone to the moduli space given in (2.51). Below (D.47) we explain the analogous procedure for $N = 4$ compactifications in the case of $d = 4$. The corresponding discussion for $d = 5$ can be found at the end of appendix E.4, below (E.53).

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[28]We thank Vicente Cortés, Paul Smyth and Antoine Van Proeyen for useful discussions on these issues.

For a chiral theory with 16 supercharges in 6 space-time dimensions the moduli space is given by $\mathcal{M} = \frac{\mathrm{SO}(5,n_T)}{\mathrm{SO}(5)\times\mathrm{SO}(n_T)}$, where n_T counts the number of tensor multiplets [33]. The gravitational multiplet contains 5 anti-self-dual antisymmetric tensors while each tensor multiplet contains a self-dual antisymmetric tensor and 5 scalars. $\mathcal{M}$ is thus spanned by the $5n_T$ scalars of the tensor multiplets.

In the superconformal framework one needs to couple 5 additional compensating tensor multiplets with 25 additional scalars. As in the non-chiral theories, the corresponding scalar manifold should be a flat cone over $\mathcal{M}$. Indeed, the ζ_I introduced in (2.61) and (2.63) define a $Gl(5)$ bundle over the moduli space (2.70). This bundle is a $5 \cdot (n_T + 5)$ dimensional flat cone over (2.70). The condition (2.64) and modding out the rotational symmetry between the ζ_I together remove these compensating multiplets and project the moduli space to (2.70).

C Generalized almost product structures

In section 3.3 we introduce a generalized almost product structure $\mathcal{P}$ on six-dimensional manifolds. In general, a generalized almost product structure may vary. However, in the following we want to show that for $\mathrm{SU}(2)\times\mathrm{SU}(2)$ structures, deformations of $\mathcal{P}$ correspond to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ doublets, which are projected out together with the massive gravitinos to achieve a consistent $N = 4$ theory (see section 3.1). In particular, we show the same result also for a conventional almost product structure P, which plays a similar role in section 3.2.

Let us start with some conventional almost product structure P and afterwards generalize the result to a generalized almost product structure $\mathcal{P}$. An almost product structure P whose eigenspace with eigenvalue $+1$ has dimension p can be described by some p-form α which is locally decomposable into one-forms. In our case we have to deal with some nowhere-vanishing two-form α which is locally decomposable, i.e. whose square is zero

$$\alpha \wedge \alpha = 0 \ . \tag{C.1}$$

The splitting of the tangent space which defines P is then obtained by

$$TY = T_2Y \oplus T_4Y \equiv A(TY) \oplus \mathrm{kernel}(A), \tag{C.2}$$

where $A = \frac{1}{2}(1 + P)$ is understood as the linear map that is related to α by some arbitrary but fixed metric.[29]

We can now try to understand the moduli space of α. Since α must stay locally decomposable under all valid deformations, α can only vary by some two-form that has one leg on each eigenspace of P, i.e. by some two-form in $T_2^*Y \wedge T_4^*Y$. For $\mathrm{SU}(2)$ structure compactifications T_4^*Y consists of $\mathrm{SU}(2)$ doublets only, while T_2^*Y consists of $\mathrm{SU}(2)$ singlets. Hence, these two-forms are doublets under the $\mathrm{SU}(2)$ structure group. Since all of these doublets are projected out together with the massive gravitino multiplets, we see that there

---

[29]We can always use the metric that corresponds to the actual point in moduli space. Therefore, the use of a metric in this argument about allowed deformations does not introduce any further assumption.

are no valid global deformations of the almost product structure. This shows that for any $SU(2)$ structure P is fixed.

Now let us consider some generalized almost product structure $\mathcal{P}$. We consider the standard form for $\mathcal{P}$ that is[30]

$$\mathcal{P}_0 = \begin{pmatrix} P & 0 \\ 0 & P^t \end{pmatrix} , \tag{C.3}$$

where P is some almost product structure on TY whose eigenspace with eigenvalue -1 has dimension four. We want to analyze its orbit under $SO(6,6)$. For this we use the decomposition of the algebra of $SO(6,6)$ given in (A.13). By $Gl(6)$ transformations we only transform P, which is already captured by the fact that we chose P to be a general almost product structure. We showed above that P can only be deformed by $SU(2)$ doublets. The remaining $SO(6,6)$ transformation of $\mathcal{P}$ are two-form and bi-vector shifts. Under a two-form shift, $\mathcal{P}$ transforms as

$$\begin{pmatrix} P & 0 \\ 0 & P^t \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \cdot \begin{pmatrix} P & 0 \\ 0 & P^t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} P & 0 \\ (B \cdot P - P^t \cdot B) & P^t \end{pmatrix} . \tag{C.4}$$

If B has both legs on $T_4^* Y$ or both on $T_2^* Y$, this transformation leaves $\mathcal{P}$ just invariant. The same result holds for bi-vector shifts. Therefore, the two-form and bi-vector shifts that change $\mathcal{P}$ have one leg on each eigenspace of P. However, these transformations are $SU(2)$ doublets. They are projected out together with the massive gravitino multiplets.

Finally, let us remark that the transformations of $SO(6,6)$ that deform $\mathcal{P}$ are those elements that are not in $SO(2,2) \times SO(4,4)$. Thus, projecting out the massive gravitinos corresponds to the projection of $SO(6,6)$ to $SO(2,2) \times SO(4,4)$.

D   Calculations in the $E_{7(7)}$ covariant formalism

The U-duality group is the symmetry group of type II supergravity, which is broken to a discrete subgroup by non-perturbative effects [43]. It naturally incorporates the T-duality group and the S-duality group which form the symmetry group of the NS sector. The $E_{7(7)}$ covariant formalism introduced in [18] is therefore a natural language to discuss the moduli space of type II theories in $d = 4$, and incorporates the pure spinor approach of generalized geometry [19, 21].

In this appendix we present the technical computations which we omitted in section 3.5 and 3.6.

D.1   Facts about $E_{7(7)}$

The goal of this appendix is to show that both (3.46) and (3.47) describe $SO(6,6) \times Sl(2,\mathbb{R})$ subgroups of $E_{7(7)}$ and to determine the decomposition of its representations in terms of the projected geometrical group $G^{\text{NS}} = Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_T \times Sl(2,\mathbb{R})_U \times SO(4,4)$, i.e. the S-duality group times the projected T-duality group.

---

[30]Note that we can always diagonalize $\mathcal{P}$ such that it reduces to some conventional almost product structure. Since the number of deformations is independent of the basis, this argument generalizes to arbitrary $\mathcal{P}$.

Let us first assemble some properties of $E_{7(7)}$ which we need later on. A more complete list of facts can be found in [21, 45]. The fundamental representation of $E_{7(7)}$ decomposes under $G^{\rm TS} = Sl(2,\mathbb{R})_S \times {\rm SO}(6,6)$ into the $(\mathbf{2},\mathbf{12})$ and the $(\mathbf{1},\mathbf{32})$ representation (see eq. (3.37)). Thus its elements can be written as

$$\lambda = (\lambda_i^A, \lambda^\pm)\,, \quad i = 1,2\,, \quad A = 1,\dots,12\,, \tag{D.1}$$

where λ_i^A is an $Sl(2,\mathbb{R})_S$ doublet of ${\rm SO}(6,6)$ vectors. $\lambda^\pm$ denotes a chiral spinor of ${\rm SO}(6,6)$ where here and in the following, the upper (lower) chirality sign is valid for type IIA (IIB). The adjoint of $E_{7(7)}$ decomposes under $Sl(2,\mathbb{R})_S \times {\rm SO}(6,6)$ into the adjoint representations of $Sl(2,\mathbb{R})_S$ and ${\rm SO}(6,6)$ and into the $(\mathbf{2},\bar{\mathbf{32}})$ representation (cf. (3.38)). Therefore its elements are of the form

$$M = (\hat{m}^i{}_j, m^A{}_B, \tilde{m}^{i\mp})\,, \tag{D.2}$$

where $\hat{m}^i{}_j$ is in the adjoint of $Sl(2,\mathbb{R})_S$, $m^A{}_B$ is in the adjoint of ${\rm SO}(6,6)$ and $\tilde{m}^{i\mp}$ is a doublet of ${\rm SO}(6,6)$ spinors. The transformation law of λ reads

$$\delta_M \lambda = \left(\hat{m}^i{}_j \lambda^{jA} + m^A{}_B \lambda^{iB} + \langle \tilde{m}^{i\mp}, \Gamma^A \lambda^\pm \rangle, \frac{1}{4} m_{AB} \Gamma^{AB} \lambda^\pm + \epsilon_{ij} \lambda^{iA} \mathcal{I}_{AB} \Gamma^B \tilde{m}^{j\mp} \right)\,. \tag{D.3}$$

The transformation law of M is given by the commutator

$$
\begin{aligned}
\delta_M N = [M,N] = \Big(& [\hat{m},\hat{n}]^i{}_j + \epsilon_{jk}\left(\langle \tilde{m}^{i\mp}, \tilde{n}^{k\mp} \rangle - \langle \tilde{n}^{i\mp}, \tilde{m}^{k\mp} \rangle \right)\,, \\
& [m,n]^A{}_B + \epsilon_{ij} \langle \tilde{m}^{i\mp}, \Gamma^A{}_B \tilde{n}^{j\mp} \rangle\,, \\
& \hat{m}^i{}_j \tilde{n}^{j\mp} - \hat{n}^i{}_j \tilde{m}^{j\mp} + \frac{1}{4}\, m_{AB} \Gamma^{AB} \tilde{n}^{i\mp} - \frac{1}{4}\, n_{AB} \Gamma^{AB} \tilde{m}^{i\mp} \Big)\,.
\end{aligned}
\tag{D.4}
$$

In the following we will restrict ourself to type IIA. For the analogous analysis in type IIB one has only to exchange some chiralities. Specifically one has to change

$$
\begin{aligned}
\Phi^+ &\longleftrightarrow \Phi^-\,, \\
C \in \Lambda^{\rm odd} T^* Y_6 &\longrightarrow C \in \Lambda^{\rm even} T^* Y_6\,, \\
Sl(2,\mathbb{R})_U &\longleftrightarrow Sl(2,\mathbb{R})_T\,, \\
\lambda^+ &\longrightarrow \lambda^-\,, \\
\tilde{m}^{i-} &\longrightarrow \tilde{m}^{i+}\,, \\
\mu^{i-} &\longrightarrow \mu^{i+}\,.
\end{aligned}
\tag{D.5}
$$

Now consider the decomposition of the adjoint representation of ${\rm SO}(6,6)$ under the breaking ${\rm SO}(6,6) \to {\rm SO}(4,4) \times {\rm SO}(2,2)$. In terms of the Lie algebras, we can further decompose $so(2,2)$ into $sl(2,\mathbb{R})_T$ and $sl(2,\mathbb{R})_U$, according to (3.34). The two sub-algebras in terms of their generators are given by

$$sl(2,\mathbb{R})_T = {\rm span}\left(\Gamma^{1+2+}, \Gamma^{1-2-}, \left(\Gamma^{1+1-} + \Gamma^{2+2-} \right) \right) \tag{D.6}$$

and

$$sl(2,\mathbb{R})_U = {\rm span}\left(\Gamma^{1+2-}, \Gamma^{2+1-}, \left(\Gamma^{1+1-} - \Gamma^{2+2-} \right) \right)\,. \tag{D.7}$$

Here, $\Gamma^{i\pm j\pm}$ denotes the anti-symmetric product of two gamma-matrices that fulfill the Clifford algebra (A.7) and which can be associated with one-forms and vectors as written in (A.10). The above decomposition can be further understood by consideration of (A.13). The $sl(2,\mathbb{R})$ sub-algebra of $gl(2,\mathbb{R})$ can be associated with $sl(2,\mathbb{R})_U$, while the generator of the global factor together with the two-form and bi-vector shifts form the $sl(2,\mathbb{R})_T$. Due to the splitting (3.34) we can consider the decomposition of the representations under the breaking $\mathrm{SO}(6,6) \to \mathrm{SO}(4,4) \times Sl(2,\mathbb{R})_T \times Sl(2,\mathbb{R})_U$. The vector representation splits as

$$\mathbf{12} \to (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{8}^v,\mathbf{1},\mathbf{1}) \,, \tag{D.8}$$

while the spinor representation decomposes as

$$\mathbf{32} \to (\mathbf{8}^s,\mathbf{2},\mathbf{1}) \oplus (\mathbf{8}^c,\mathbf{1},\mathbf{2}) \,. \tag{D.9}$$

The adjoint decomposes as

$$\mathbf{66} \to (\mathbf{28},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{8}^v,\mathbf{2},\mathbf{2}) \,. \tag{D.10}$$

Now we have everything to show that (3.46) and (3.47) both describe $\mathrm{SO}(6,6) \times Sl(2,\mathbb{R})$ subgroups of $E_{7(7)}$. Let us start with the type IIA case, and consider the sub-algebra (3.46) of the adjoint representation (3.41) of $E_{7(7)}$. Analogously to (3.46), we write an element in g^{IIA} as

$$M = \left(\hat{m}^i{}_j, m^p{}_q, m^a{}_b, \tilde{m}^{i-} \right) \,, \tag{D.11}$$

where we have decomposed the $m^A{}_B$ of eq. (D.2) into the $\mathrm{SO}(2,2)$ generators $m^p{}_q, p,q = 1,\dots,4$ and the $\mathrm{SO}(4,4)$ generators $m^a{}_b, a,b = 1,\dots,8$. Finally, we note that $\tilde{m}^{i-} \in \Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6$.

We can determine the commutator of two elements in g^{IIA} by restricting (D.4) to elements in g^{IIA}. We obtain

$$
\begin{aligned}
[M,N] = &\Big([\hat{m},\hat{n}]^i{}_j + \epsilon_{jk} \left(\langle \tilde{m}^{i-}, \tilde{n}^{k-} \rangle + \langle \tilde{m}^{k-}, \tilde{n}^{i-} \rangle \right), \ [m,n]^p{}_q + \epsilon_{ij} \langle \tilde{m}^{i-}, \Gamma^p{}_q \tilde{n}^{j-} \rangle, \\
&[m,n]^a{}_b + \epsilon_{ij} \langle \tilde{m}^{i-}, \Gamma^a{}_b \tilde{n}^{j-} \rangle, \ \hat{m}^i{}_j \tilde{n}^{j-} - \hat{n}^i{}_j \tilde{m}^{j-} \\
&+ \frac{1}{4}\, m_{ab} \Gamma^{ab} \tilde{n}^{i-} - \frac{1}{4}\, n_{ab} \Gamma^{ab} \tilde{m}^{i-} + \frac{1}{4}\, m_{pq} \Gamma^{pq} \tilde{n}^{i-} - \frac{1}{4}\, n_{pq} \Gamma^{pq} \tilde{m}^{i-} \Big) \,.
\end{aligned}
\tag{D.12}
$$

As explained above, we can further decompose the above expression due to (3.34). However, because $\tilde{m}^{i-}$ and $\tilde{n}^{i-}$ live in $\Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6$, their components on $T_2 Y_6 \oplus T_2^* Y_6$ are of odd chirality. Hence, they are singlets under $Sl(2,\mathbb{R})_T$ and doublets under $Sl(2,\mathbb{R})_U$, as can be seen from (D.6) and (D.7). From (D.12) we then obtain that elements in $Sl(2,\mathbb{R})_T$ commute with all other generators of G^{IIA}, i.e. the group G^{IIA} is a direct product with one factor being $Sl(2,\mathbb{R})_T$. Furthermore, from (D.12) we see that $\tilde{m}^{i-}$ is in the $(\mathbf{8},\mathbf{2},\mathbf{2})$ of $\mathrm{SO}(4,4) \times Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U$, and from (D.10) we see that together with the remaining generators of G^{IIA} they indeed span the algebra of $\mathrm{SO}(6,6)$. Thus we conclude that $G^{\mathrm{IIA}} \equiv \mathrm{SO}(6,6) \times Sl(2,\mathbb{R})_T$.

For type IIB, the arguments are very similar, apart from the fact that in (D.11) the negative chirality spinor $\tilde{m}^{i-}$ is replaced by the positive chirality spinor $\tilde{m}^{i+}$ which lives in

$\Lambda^{\text{even}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$. Therefore, instead of $Sl(2,\mathbb{R})_T$ the factor $Sl(2,\mathbb{R})_U$ drops out and we obtain $G^{\text{IIB}} \equiv \text{SO}(6,6) \times Sl(2,\mathbb{R})_U$.

Thus, we identified three subgroups of $E_{7(7)}$. First there is $G^{\text{TS}} = Sl(2,\mathbb{R})_S \times \text{SO}(6,6)$ which arises by turning off all RR-degrees of freedom. The other two subgroups arise in $N = 4$ compactifications after projecting out all $\text{SU}(2) \times \text{SU}(2)$ doublets or equivalently all massive gravitino multiplets. They are $G^{\text{IIA}} = Sl(2,\mathbb{R})_T \times \text{SO}(6,6)$ and $G^{\text{IIB}} = Sl(2,\mathbb{R})_U \times \text{SO}(6,6)$. The common subgroup of all three groups is $G^{\text{NS}} = Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_T \times Sl(2,\mathbb{R})_U \times \text{SO}(4,4)$.

It is instructive to decompose the $E_{7(7)}$ representations with respect to the breaking $E_{7(7)} \rightarrow G^{\text{TS}} \rightarrow G^{\text{NS}}$. Using (3.37) together with (D.8) and (D.9), the fundamental representation decomposes into

$$
\begin{aligned}
\mathbf{56} \rightarrow &(\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}) \\
\rightarrow &(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8}^v) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8}^s) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8}^c),
\end{aligned}
\tag{D.13}
$$

and similarly using (D.10) and (D.9) the adjoint (3.38) breaks into

$$
\begin{aligned}
\mathbf{133} \rightarrow &(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\bar{\mathbf{32}}) \\
\rightarrow &(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{28}) \\
&\oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{8}^v) \oplus (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{8}^s) \oplus (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{8}^c).
\end{aligned}
\tag{D.14}
$$

This shows that both (D.13) and (D.14) are invariant under the joint action of $\text{SO}(4,4)$ triality together with the corresponding STU triality action interchanging the three $Sl(2,\mathbb{R})$ factors.

The part of the triality action interchanging T and U just corresponds to a generalization of T-duality between type IIA and type IIB. However, the interchange of S with T or U seems to lead to some theory where the RR-sector is absent but no $\text{SU}(2) \times \text{SU}(2)$ doublets have been projected out. For type II compactifications on $K3 \times T^2$, this corresponds to the non-perturbative duality with the heterotic string compactified on T^6. In this case, the triality action is known as string/string/string triality [46]. The heterotic duals of some specific $\text{SU}(2)$ structure compactifications have been analyzed in [23].

D.2 Calculating stabilizers and orbits

Let us now compute the stabilizer of the pure $\text{SO}(6,6)$ spinors embedded in the fundamental and adjoint representation of $E_{7(7)}$. We consider type IIA compactifications in some detail and only give the results for type IIB which can easily be obtained by some chirality changes, as explained in appendix D.1. Furthermore we use the reduction $E_{7(7)} \rightarrow G^{\text{IIA}} = Sl(2,\mathbb{R})_T \times \text{SO}(6,6)$ and compute the intersection of the stabilizer with this subgroup. Note that we only have to consider the real or the imaginary part of some pure $\text{SO}(6,6)$ spinor because they are related via the Hitchin functional [13].

D.2.1 Embedding into the fundamental

Let us start with the pure spinor Φ^+ of positive chirality. According to (3.37), we can embed its real part $\text{Re}(\Phi^+)$ into the fundamental representation as done in (3.48). We

can compute the stabilizer using (D.3) and the splitting (3.46). We know that $\operatorname{Re}\Phi^+$ is a singlet under $Sl(2,\mathbb{R})_S$ and, since $\operatorname{Re}\Phi^+ \in \Lambda^{\text{even}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$, also a singlet under $Sl(2,\mathbb{R})_U$. Therefore both groups are in the stabilizer. Due to the same reason, $\operatorname{Re}\Phi^+$ transforms under $Sl(2,\mathbb{R})_T$ non-trivially such that no element in this group stabilizes $\operatorname{Re}\Phi^+$. Furthermore, Φ_1 defined in (3.30) is pure and is therefore stabilized by $\text{SU}(2,2)_{\Phi_1} \equiv \text{SO}(2,4) \subset \text{SO}(4,4)$. Therefore, $\operatorname{Re}\Phi^+$ is also stabilized by this subgroup.

It remains to understand which elements $\tilde{m}^{i-} \in \Lambda^{\text{odd}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$ stabilize λ. We use the parameterization (3.30) and (3.32). Thus, $\operatorname{Re}\Phi^+$ can be expressed as[31]

$$\operatorname{Re}\Phi^+ = 1 \wedge \operatorname{Re}\Phi_1 - \operatorname{Re}K \wedge \operatorname{Im}K \wedge \operatorname{Im}\Phi_1\,, \tag{D.15}$$

where K is the holomorphic one-form on $\Lambda^{\text{odd}}T_2^*Y_6$ defined in (3.8) and related to the pure spinors in (3.32). The transformation parameters can be analogously written as

$$\tilde{m}^{i-} = \operatorname{Re}K \wedge \alpha^{i1} + \operatorname{Im}K \wedge \alpha^{i2}\,, \quad i = 1, 2\,, \tag{D.16}$$

where α^{ij} are spinors in $\Lambda^{\text{even}}T_4^*Y_6$

From (D.3) we now see that those $\tilde{m}^{i-}$ are in the stabilizer which fulfill

$$\langle \tilde{m}^{i-}, \Gamma^A \operatorname{Re}\Phi^+ \rangle = 0 \qquad \text{for all} \quad A = 1, \ldots, 12\,. \tag{D.17}$$

These equations are only non-trivial for the gamma-matrices that act on $\Lambda^{\text{odd}}T_2^*Y_6$. Thus, we arrive at

$$\langle \alpha^{ij}, \operatorname{Re}\Phi_1 \rangle = \langle \alpha^{ij}, \operatorname{Im}\Phi_1 \rangle = 0\,. \tag{D.18}$$

This just eliminates all $SU(2,2)_{\Phi_1}$ singlets. Therefore, the elements in $\Lambda^{\text{odd}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$ which are in the stabilizer just form the representation $(\mathbf{2}, \mathbf{2}, \mathbf{6})$ of the group $Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U \times SU(2,2)_{\Phi_1}$. This combines with the adjoint of $Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U \times SU(2,2)_{\Phi_1}$ into the adjoint of $\text{SO}(4,6)$. There is one further $\text{SO}(2)$ factor in the stabilizer which corresponds to a simultaneous rotation of the doublets $(\operatorname{Re}\Phi_1, \operatorname{Im}\Phi_1)$ and $(1, \operatorname{Re}K \wedge \operatorname{Im}K)$ such that the overall phase factor of Φ^+ stays constant. Therefore, the whole stabilizer is $\text{SO}(4,6) \times \text{SO}(2)$. This can also be understood from the fact that $\operatorname{Re}\Phi^+$ transforms non-trivially under $Sl(2,\mathbb{R})_T$ and therefore λ must be in the $(\mathbf{2}, \mathbf{12})$ of G^{IIA}. This just gives a doublet of space-like $\text{SO}(6,6)$ spinors, which is indeed stabilized by $\text{SO}(4,6) \times \text{SO}(2)$.

In order to obtain the physical parameter space we need to mod out by one further gauge degree of freedom which corresponds to the phase of Φ^+, as explained in section 3.6. We thus finally get

$$\mathcal{M}_\lambda = \frac{\text{SO}(6,6)}{\text{SO}(2) \times \text{SO}(4,6)} \times \frac{Sl(2,\mathbb{R})_T}{\text{SO}(2)}\,. \tag{D.19}$$

The first factor just describes a space-like two-dimensional subspace in a space of signature $(6,6)$.

Before we move on to the embedding of the other spinor, we want to know what is the most general form for the embedding of λ given in (3.48) if it is deformed by some

---

[31]For simplicity we again switched off the B-field which, however, does not change the computation.

general G^{IIA} deformation. The only deformations that are not already caught by the Ansatz in (3.48) are the elements in the adjoint of G^{IIA} that are represented by $\Lambda^{\text{odd}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$. We just showed that the part of $\Lambda^{\text{odd}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$ which is in the $(\mathbf{2},\mathbf{2},\mathbf{6})$ representation of the group $Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U \times SU(2,2)_{\Phi_1}$ stabilizes λ, and therefore cannot deform it. Thus, only the deformations in the $(\mathbf{2},\mathbf{2},\mathbf{1})$ representation, i.e. $SU(2,2)_{\Phi_1}$ singlets, may deform λ. These are spinors of the form

$$m^{i-} = (c^{i2+}\operatorname{Re}K - c^{i1+}\operatorname{Im}K) \wedge \operatorname{Re}\Phi_1 - (c^{i1-}\operatorname{Re}K + c^{i2-}\operatorname{Im}K) \wedge \operatorname{Im}\Phi_1 , \qquad \text{(D.20)}$$

where the $c^{ia}, a = 1+, 1-, 2+, 2-$, are real. The action of such elements $\tilde{m}^{i-}$ in the adjoint on λ reads

$$\delta_{\tilde{m}^{i-}}\lambda = \left(c^{ia}, 0\right) , \qquad \text{(D.21)}$$

where we used (3.48), (D.15) and (D.3). Note that c^{ia} is an SO(2,2) vector which naturally embeds into an SO(6,6) vector.

By exponentiation of the m^{i-} in (D.20) the most general form of the embedding (3.48) can be obtained. These additional degrees of freedom correspond to the RR-fields and to gauge degrees of freedom. We discuss their physical significance in appendix D.2.3 when we consider both the embeddings of Φ^+ and Φ^-.

D.2.2 Embedding into the adjoint

Let us now come to the embedding of the other pure spinor Φ^-. Due to its negative chirality Φ^- cannot be embedded into the fundamental representation of $E_{7(7)}$ but only into its adjoint via $\mu_1 = (0, 0, \operatorname{Re}(u^i\Phi^-))$ as we argued in (3.52).[32] Now let us analyze the intersection of its stabilizer with G^{IIA}. Since $E_{7(7)}$ acts on μ_1 via the Lie bracket, we can use (D.12) to determine its stabilizer. With help of (3.30) and (3.32) we can write it as $\mu_1 = (0, 0, \operatorname{Re}(u^iK \wedge \Phi_2))$. One can easily see that $Sl(2,\mathbb{R})_S$ acts freely on μ_1 since it acts freely on u^i. Since $\operatorname{Re}\Phi^-, \operatorname{Im}\Phi^- \in \Lambda_2^{\text{odd}}T^*Y_6 \wedge \Lambda_4^{\text{even}}T^*Y_6$, we also see that $Sl(2,\mathbb{R})_U$ acts freely on μ_1. However, there are two additional phase rotations inside $Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U \times SO(4,4)$ that leave μ_1 invariant. The first rotates u^i and the spinor component $K \in \Lambda_2^{\text{odd}}T^*Y_6$ of Φ^- with opposite phases. We call the generator of this transformation $R_{(+1,-1,0)}$. The second generator, $R_{(+1,+1,-2)}$, rotates u^i and K by the same phase and Φ_2 oppositely.

$Sl(2,\mathbb{R})_T$ acts trivially and therefore is part of the stabilizer of μ_1. Since Φ_2 is pure, we know that it is also stabilized by an $SU(2,2)_{\Phi_2} \subset SO(4,4)$ subgroup, and so is Φ^-. It remains to determine the elements $\tilde{m}^{i-}$ of $\Lambda^{\text{odd}}T_2^*Y_6 \wedge \Lambda^{\text{even}}T_4^*Y_6$ which leave μ_1 invariant. From (D.12) we see that this leads to the following equations

$$\begin{aligned} \langle \tilde{m}^{i-}, \operatorname{Re}\left(u^j\Phi^-\right)\rangle + \langle \tilde{m}^{j-}, \operatorname{Re}\left(u^i\Phi^-\right)\rangle &= 0, \\ \epsilon_{ij}\langle \tilde{m}^{i-}, \Gamma^p{}_q \operatorname{Re}\left(u^j\Phi^-\right)\rangle &= 0, \\ \epsilon_{ij}\langle \tilde{m}^{i-}, \Gamma^a{}_b \operatorname{Re}\left(u^j\Phi^-\right)\rangle &= 0 . \end{aligned} \qquad \text{(D.22)}$$

---

[32]Note that we distinguish in our notation formally between M and μ_1 which are both elements in the adjoint of $E_{7(7)}$. The former one corresponds to an unspecified generator of the group action. The latter one parameterizes the $SU(2) \times SU(2)$ structure and is defined by the embedding of the pure spinor Φ^-.

For convenience, we choose $u^1 = 1$ and $u^2 = -\mathrm{i}$ and insert (D.16) into the above equations. Using $\operatorname{Re} \Phi^- = \operatorname{Re} K \wedge \operatorname{Re} \Phi_2 - \operatorname{Im} K \wedge \operatorname{Im} \Phi_2$ and $\operatorname{Im} \Phi^- = \operatorname{Re} K \wedge \operatorname{Im} \Phi_2 + \operatorname{Im} K \wedge \operatorname{Re} \Phi_2$, the first equation in (D.22) implies

$$\langle \alpha^{11}, \operatorname{Im} \Phi_2 \rangle = -\langle \alpha^{12}, \operatorname{Re} \Phi_2 \rangle \,,$$
$$\langle \alpha^{22}, \operatorname{Im} \Phi_2 \rangle = \langle \alpha^{21}, \operatorname{Re} \Phi_2 \rangle \,, \tag{D.23}$$
$$\langle \alpha^{11}, \operatorname{Re} \Phi_2 \rangle - \langle \alpha^{12}, \operatorname{Im} \Phi_2 \rangle = \langle \alpha^{22}, \operatorname{Re} \Phi_2 \rangle + \langle \alpha^{21}, \operatorname{Im} \Phi_2 \rangle \,.$$

Similarly, the second equation in (D.22) leads to

$$\langle \alpha^{11}, \operatorname{Im} \Phi_2 \rangle = \langle \alpha^{21}, \operatorname{Re} \Phi_2 \rangle \,,$$
$$\langle \alpha^{22}, \operatorname{Im} \Phi_2 \rangle = -\langle \alpha^{12}, \operatorname{Re} \Phi_2 \rangle \,, \tag{D.24}$$
$$\langle \alpha^{11}, \operatorname{Re} \Phi_2 \rangle + \langle \alpha^{12}, \operatorname{Im} \Phi_2 \rangle = \langle \alpha^{22}, \operatorname{Re} \Phi_2 \rangle - \langle \alpha^{21}, \operatorname{Im} \Phi_2 \rangle \,.$$

Together, they give five relations between the eight $SU(2,2)_{\Phi_2}$ singlets, which read

$$\langle \alpha^{11}, \operatorname{Im} \Phi_2 \rangle = -\langle \alpha^{12}, \operatorname{Re} \Phi_2 \rangle = \langle \alpha^{21}, \operatorname{Re} \Phi_2 \rangle = \langle \alpha^{22}, \operatorname{Im} \Phi_2 \rangle \,,$$
$$\langle \alpha^{11}, \operatorname{Re} \Phi_2 \rangle = \langle \alpha^{22}, \operatorname{Re} \Phi_2 \rangle \,, \tag{D.25}$$
$$\langle \alpha^{12}, \operatorname{Im} \Phi_2 \rangle = -\langle \alpha^{21}, \operatorname{Im} \Phi_2 \rangle \,.$$

This means that only three of the $SU(2,2)_{\Phi_2}$ singlets are elements in the stabilizer. It remains to analyze the third equation in (D.22). By the same method, we can write it as

$$\langle \alpha^{11}, \Gamma^a{}_b \operatorname{Re} \Phi_2 \rangle - \langle \alpha^{12}, \Gamma^a{}_b \operatorname{Im} \Phi_2 \rangle + \langle \alpha^{21}, \Gamma^a{}_b \operatorname{Im} \Phi_2 \rangle + \langle \alpha^{22}, \Gamma^a{}_b \operatorname{Re} \Phi_2 \rangle = 0 \,, \tag{D.26}$$

where a and b are arbitrary $SO(4,4)$ indices. For the singlets these equations imply additionally

$$\langle \alpha^{11}, \operatorname{Re} \Phi_2 \rangle - \langle \alpha^{12}, \operatorname{Im} \Phi_2 \rangle + \langle \alpha^{21}, \operatorname{Im} \Phi_2 \rangle + \langle \alpha^{22}, \operatorname{Re} \Phi_2 \rangle = 0 \,. \tag{D.27}$$

Together with the third equation in (D.23) this implies

$$\langle \alpha^{11}, \operatorname{Re} \Phi_2 \rangle = \langle \alpha^{12}, \operatorname{Im} \Phi_2 \rangle \,, \tag{D.28}$$

which reduces the number of singlets to two.

In eq. (D.26) we can choose $\Gamma^a{}_b \operatorname{Re} \Phi_2$ to be in the $\mathbf{6}$ of $SU(2,2)_{\Phi_2}$ and $\Gamma^a{}_b \operatorname{Im} \Phi_2 = 0$ or the other way around. This implies

$$\alpha \equiv \operatorname{Proj}_{\mathbf{6}} \alpha^{11} = -\operatorname{Proj}_{\mathbf{6}} \alpha^{22} \,, \qquad \beta \equiv \operatorname{Proj}_{\mathbf{6}} \alpha^{12} = \operatorname{Proj}_{\mathbf{6}} \alpha^{21} \,, \tag{D.29}$$

where $\operatorname{Proj}_{\mathbf{6}}$ is the projection onto the $\mathbf{6}$ representations of $SU(2,2)_{\Phi_2}$. This gives twelve conditions on the remaining 24 degrees of freedom and eliminates two of the four $\mathbf{6}$ representations of $SU(2,2)_{\Phi_2}$. Therefore, we can parameterize the m^{i-} in the stabilizer by

$$\tilde{m}_{\mathbf{6}}^{i-} = \begin{pmatrix} \operatorname{Re} K \wedge \alpha + \operatorname{Im} K \wedge \beta \\ \operatorname{Re} K \wedge \beta - \operatorname{Im} K \wedge \alpha \end{pmatrix} \,, \tag{D.30}$$

and

$$\tilde{m}_{\mathbf{1}}^{i-} = \begin{pmatrix} \operatorname{Re} K \wedge (a \operatorname{Re} \Phi_2 + b \operatorname{Im} \Phi_2) - \operatorname{Im} K \wedge (b \operatorname{Re} \Phi_2 - a \operatorname{Im} \Phi_2) \\ \operatorname{Re} K \wedge (b \operatorname{Re} \Phi_2 - a \operatorname{Im} \Phi_2) + \operatorname{Im} K \wedge (a \operatorname{Re} \Phi_2 + b \operatorname{Im} \Phi_2) \end{pmatrix}, \qquad \text{(D.31)}$$

where $a, b \in \mathbb{R}$. We can reparameterize (D.30) and (D.31) in a more elegant way by writing

$$\tilde{m}_{\mathbf{6}}^{i-} = \operatorname{Re}(u^i \bar{K} \wedge \alpha_{\mathbb{C}}), \qquad \tilde{m}_{\mathbf{1}}^{i-} = \operatorname{Re}(c u^i K \wedge \bar{\Phi}_2), \qquad \text{(D.32)}$$

where $\alpha_{\mathbb{C}}$ is in the complexified $\mathbf{6}$ representation of $SU(2,2)_{\Phi_2}$ and $c \in \mathbb{C}$. One can check that the algebra of the transformations in the $\mathbf{6}$ representation closes if one includes the $SO(2) \subset Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U$ factor $R_{(+1,-1,0)}$ that is part of the stabilizer as well as the $SU(2,2)_{\Phi_2} \equiv SO(2,4) \subset SO(4,4)$. Together they form the group $SO(2,6)$. The algebra of the $SU(2,2)_{\Phi_2}$ singlet transformations $\tilde{m}_{\mathbf{1}}^{i-}$ does close if one includes the transformation $R_{(+1,+1,-2)}$ we introduced above. Together, they generate some $SO(3)$ group and therefore the complete stabilizer is $SO(3) \times SO(2,6)$.

However, there are still some gauge transformations of μ_1 that do not have any physical meaning and must be removed. In section 3.6 we explain that μ_1 is part of some highest weight $SU(2)$ embedding μ_a with an $SU(2)$ gauge freedom which is generated by the μ_a themselves. The generator $R_{(+1,+1,+2)}$ coincides with μ_3, which is defined in (3.54), and rotates μ_1 and μ_2 defined in (3.53) into each other. Furthermore, μ_1 and μ_2 are the other two $SU(2)$ generators $\tilde{m}_{\mathbf{1}}^{i-}$ that correspond to gauge freedom. These three generators form the algebra of $su(2)$, and furthermore commute with the stabilizer $su(2) = so(3)$ on the singlets which we discussed above. Therefore, all of them together form the algebra of $so(4) = su(2) \oplus su(2)$, explicitly spanned by $R_{(+1,+1,+2)}$, $R_{(+1,+1,-2)}$ and

$$\begin{aligned} \tilde{m}_{\mathbf{1}}^{i-} &= \begin{pmatrix} \operatorname{Re} K \wedge (a \operatorname{Re} \Phi_2 + d \operatorname{Im} \Phi_2) - \operatorname{Im} K \wedge (b \operatorname{Re} \Phi_2 + c \operatorname{Im} \Phi_2) \\ \operatorname{Re} K \wedge (b \operatorname{Re} \Phi_2 + c \operatorname{Im} \Phi_2) + \operatorname{Im} K \wedge (a \operatorname{Re} \Phi_2 + d \operatorname{Im} \Phi_2) \end{pmatrix} \\ &= \operatorname{Re}(c_1 u^i K \wedge \bar{\Phi}_2 + c_2 u^i K \wedge \Phi_2). \end{aligned} \qquad \text{(D.33)}$$

Here, a, b, c and d are real parameters that can be rewritten in terms of the complex parameters c_1 and c_2. The generators corresponding to c_1 are the elements in the stabilizer we already had in (D.32).

As explained in section 3.6, the stabilizer of μ_1 coincides with the stabilizer of the other two μ_a. Therefore, after removing pure gauge degrees of freedoms, we end up with the configuration space

$$\mathcal{M}_{\mu_a} = \frac{SO(6,6)}{SO(4) \times SO(2,6)}. \qquad \text{(D.34)}$$

We see that μ_1 defines up to gauge equivalence a space-like four-dimensional subspace in a space of signature $(6,6)$. This can be understood as follows. Under the breaking $SO(6,6) \to Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_U \times SO(4,4)$, the vector representation of $SO(6,6)$ decomposes analogously to (D.8) as $\mathbf{12} \to (\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{8}^s)$, where we used the triality of $SO(4,4)$. Under the same decomposition, μ sits in the $(\mathbf{2}, \mathbf{2}, \mathbf{8}^s)$ representation which is just the tensor product $(\mathbf{2}, \mathbf{2}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{1}, \mathbf{8}^s)$. The first factor in this tensor product is given

by $u^i K$ while the second is given by Φ_2. Both define complex $SO(6,6)$ vectors which are orthogonal to each other. From the normalization in (3.10) and (3.51) we see that[33]

$$\epsilon_{ij}\langle u^i K, \bar{u}^j \bar{K}\rangle_2 = 2 \ . \tag{D.35}$$

Thus, we can conclude that the real and imaginary part of $u^i K$ have positive norm. The same holds for the real and imaginary part of Φ_2. Hence, together they span the space-like four-plane defined by μ (or equivalently, by the μ_a) in the space of signature $(6,6)$. Note that the $SO(4)$ transformations $R_{(+1,+1,+2)}$, $R_{(+1,+1,-2)}$ and the ones given in (D.33) are really those that rotate the space-like four-plane non-trivially into itself.

Now let us determine the most general form of the embedding (3.52). For this, we consider the possible deformations of μ_1 under $G^{\mathrm{IIA}} = Sl(2,\mathbb{R})_T \times SO(6,6)$ transformations. The transformations in $G^{\mathrm{NS}} = Sl(2,\mathbb{R})_S \times Sl(2,\mathbb{R})_T \times Sl(2,\mathbb{R})_U \times SO(4,4)$ act only on u^i and Φ^-. Thus, they do not change the embedding (3.52). It remains to understand the transformation properties under $SO(6,6)$ deformations $\tilde{n}^{i-}$ which live in $\Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6$. From (D.30) and (D.33) we know the $\tilde{n}^{i-}$ that stabilize μ. This means that those which deform μ_1 non-trivially are of the form

$$\begin{aligned}\tilde{n}_{\mathbf{1}}^{i-} &= \begin{pmatrix} \operatorname{Re} K \wedge (a \operatorname{Re} \Phi_2 + d \operatorname{Im} \Phi_2) + \operatorname{Im} K \wedge (b \operatorname{Re} \Phi_2 + c \operatorname{Im} \Phi_2) \\ \operatorname{Re} K \wedge (b \operatorname{Re} \Phi_2 + c \operatorname{Im} \Phi_2) - \operatorname{Im} K \wedge (a \operatorname{Re} \Phi_2 + d \operatorname{Im} \Phi_2) \end{pmatrix} \\ &= \operatorname{Re}(d_1 u^i \bar{K} \wedge \bar{\Phi}_2 + d_2 u^i \bar{K} \wedge \Phi_2)\end{aligned} \tag{D.36}$$

for the $SU(2,2)_{\Phi_2}$ singlets, and

$$\tilde{n}_{\mathbf{6}}^{i-} = \begin{pmatrix} -\operatorname{Re} K \wedge \alpha + \operatorname{Im} K \wedge \beta \\ -\operatorname{Re} K \wedge \beta - \operatorname{Im} K \wedge \alpha \end{pmatrix} = \operatorname{Re}(u^i K \wedge \beta_{\mathbb{C}}) \tag{D.37}$$

for the $\mathbf{6}$ representation of $SU(2,2)_{\Phi_2}$. The coefficients a, b, c and d again are real and can be expressed in the complex numbers d_1 and d_2. Analogously, α and β are real $SO(4,4)$ spinors in the $\mathbf{6}$ representation of $SU(2,2)$, while $\beta_{\mathbb{C}}$ is in the complexified $\mathbf{6}$ representation.

In order to compute the transformations of μ, we decompose analogously to (D.11)

$$\delta\mu = \left((\delta\hat{\mu})^i{}_j, (\delta\mu)^p{}_q, (\delta\mu)^a{}_b, (\delta\mu)^{i-}\right) \ . \tag{D.38}$$

Using (D.12) we find

$$(\delta_{\mathbf{6}}\mu)^a{}_b = 2(\langle \alpha, \Gamma^a{}_b \operatorname{Re} \Phi_2\rangle + \langle \beta, \Gamma^a{}_b \operatorname{Im} \Phi_2\rangle) = 2\operatorname{Re}(\langle \beta_{\mathbb{C}}, \Gamma^a{}_b \bar{\Phi}_2\rangle) \tag{D.39}$$

as the only non-vanishing deformation for the deformations in the $\mathbf{6}$ representation. For the deformations in the $\mathbf{1}$ representation the variation in the adjoint of $Sl(2,\mathbb{R})$ is

$$(\delta_{\mathbf{1}}\hat{\mu})^i{}_j = 2 \begin{pmatrix} a-c & b+d \\ b+d & c-a \end{pmatrix} = 2\operatorname{Im}(d_1 \epsilon_{jk} u^i u^k) \ . \tag{D.40}$$

---

[33]Here, $\langle \cdot, \cdot \rangle_2$ is the usual spinor product on $\Lambda^{\bullet} T_2^* Y_6$ that is related to the Mukai pairing by a two-dimensional volume factor.

The deformations $(\delta_{\mathbf{1}}\mu)^p{}_q$ that take values in the adjoint of SO(2,2) do not fill it out completely but only the $sl(2,\mathbb{R})_U$ part. They read

$$(\delta_{\mathbf{1}}\mu)^p{}_q = \begin{pmatrix} a+c & d-b & 0 & 0 \\ d-b & -a-c & 0 & 0 \\ 0 & 0 & -a-c & b-d \\ 0 & 0 & b-d & a+c \end{pmatrix} = \begin{pmatrix} \mathrm{Im}(d_2\epsilon_{jk}\bar{u}^i\bar{u}^k) & 0 \\ 0 & -\mathrm{Im}(d_2\epsilon_{jk}\bar{u}^i\bar{u}^k) \end{pmatrix} , \quad (\text{D.41})$$

with all other components being zero.

D.2.3 Putting the parts together

So far we just embedded the two pure SO(6,6) spinors into appropriate $E_{7(7)}$ representations. In this section we finally discuss their purity and compatibility conditions as well as the moduli space spanned by both objects.

As we already mentioned in section 3.6 a pure (complex) SO(6,6) spinor Φ is equivalent to a stable (real) SO(6,6) spinor $\chi = \mathrm{Re}\,\Phi$. Stability means in this context that such a spinor χ transforms in an open orbit under the action of SO(6,6). Stability is measured by the quartic SO(6,6) invariant [6, 13]

$$q(\chi) = -\frac{1}{4}\,\bar{\chi}\Gamma_{AB}\chi\,\bar{\chi}\Gamma^{AB}\chi = \frac{1}{4}\,\mathrm{tr}(\mathcal{J}_\Phi^2) . \qquad (\text{D.42})$$

For χ to be stable we need $q(\chi) < 0$. Furthermore, q measures the normalization of the generalized almost complex structure $\mathcal{J}_\Phi$. Thus, $\mathcal{J}_\Phi$ can be properly normalized if $q(\mathrm{Re}\,\Phi) < 0$. The quantity (D.42) is naturally embedded into the quartic invariants of the $E_{7(7)}$ representations [21]

$$q(\lambda) = \epsilon_{ij}\epsilon_{kl}\mathcal{I}_A B \mathcal{I}_C D \lambda^{iA}\lambda^{kB}\lambda^{jC}\lambda^{lD} - \epsilon_{ij}\lambda^{iA}\lambda^{jB}\bar{\lambda}^+\Gamma_{AB}\lambda^+ - \frac{1}{24}\bar{\lambda}^+\Gamma_{AB}\lambda^+\bar{\lambda}^+\Gamma^{AB}\lambda^+ \qquad (\text{D.43})$$

and

$$q(m) = \mathrm{tr}\,m^4 + (\mathrm{tr}\,m^2)^2 + (\det\hat{m})^2 + \mathrm{tr}\,m^2\epsilon_{ij}\bar{\tilde{m}}^{i-}\tilde{m}^{j-} + (\det\hat{m})^2\epsilon_{ij}\bar{\tilde{m}}^{i-}\tilde{m}^{j-}$$
$$+ (\epsilon_{ij}\bar{\tilde{m}}^{i-}\tilde{m}^{j-})^2 + \epsilon_{ij}\epsilon_{kl}\bar{\tilde{m}}^{i-}\Gamma^{AB}\tilde{m}^{k-}\bar{\tilde{m}}^{j-}\Gamma_{AB}\tilde{m}^{l-} . \qquad (\text{D.44})$$

We see that in both expressions the last term just gives the Hitchin functional while the other terms vanish for the embeddings (3.48) and (3.52). Hence, both quartic invariants just generalize the corresponding Hitchin functionals and we impose the stability condition via

$$q(\lambda) < 0, \qquad q(\mu_1) < 0 . \qquad (\text{D.45})$$

Now we come to the issue of compatibility. By studying (D.3) one can check that (3.59) is appropriate to reproduce the SO(6,6) compatibility conditions given in (3.17) for the embeddings (3.48) and (3.52). Furthermore, in principle both embeddings (3.48) and (3.52) could each be shifted by independent $E_{7(7)}$ transformations. Equation (3.59) ensures that the additional degrees of freedom which arise in the U-duality group are the same for both embeddings.

Finally, we want to determine the parameter space of compatible μ and λ. To do so, we first determine the intersection of the two stabilizers, and afterwards eliminate all gauge redundancies. Let us distinguish for the following analysis $\mathrm{SU}(2) \times \mathrm{SU}(2)$ singlets from the $(\mathbf{2}, \mathbf{2})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The former one is spanned by the real and imaginary parts of Φ_1 and Φ_2. The latter one is just the intersection of the $\mathbf{6}$ representations of $SU(2,2)_{\Phi_1}$ and $SU(2,2)_{\Phi_2}$.

Let us start with the $(\mathbf{2}, \mathbf{2})$ representation. The part of the common stabilizer that is in $so(4,4)$ is just $su(2) \times su(2) \equiv so(4)$, as we know from the theory of pure compatible spinors. It just describes rotations in the space orthogonal to the real and imaginary parts of the two pure spinors and therefore is in the $(\mathbf{2}, \mathbf{2})$ representation. Furthermore, there are additional elements in the $(\mathbf{2}, \mathbf{2})$ representation of the stabilizer which are of the form $\tilde{m}^{i-}$. The $\tilde{m}^{i-}$ in (D.30) split into those that are in the $(\mathbf{2}, \mathbf{2})$ representation and those which are $\mathrm{SU}(2) \times \mathrm{SU}(2)$ singlets. Eq. (D.18) eliminates exactly the singlets. The algebra consisting of the $\tilde{m}^{i-}$ in the $(\mathbf{2}, \mathbf{2})$ representation and the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ generators closes if one includes the $\mathrm{SO}(2)$ factor $R_{(+1,-1,0)}$. Analogously to the analysis for the stabilizer of μ, these transformations form the group $\mathrm{SO}(6)$.

Now let us consider the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ singlets. These singlets are either $SU(2,2)_{\Phi_1}$ or $SU(2,2)_{\Phi_2}$ singlets. The singlets of $SU(2,2)_{\Phi_1}$ are removed from the stabilizer by (D.18). The $SU(2,2)_{\Phi_2}$ singlets all stabilize λ. Hence, the singlet component of the stabilizer is the same as the one for μ. In section D.2.2, we showed that it forms the group $\mathrm{SO}(3)$. Again there are further gauge redundancies which have to be projected out. Part of them we already discussed in the previous two subsections. Furthermore, the gauge transformation that rotate the doublets $(\operatorname{Re}\Phi_1, \operatorname{Im}\Phi_1)$ and $(\operatorname{Re}\Phi_2, \operatorname{Im}\Phi_2)$ are part of a bigger group that rotates the vector $(\operatorname{Re}\Phi_1, \operatorname{Im}\Phi_1, \operatorname{Re}\Phi_2, \operatorname{Im}\Phi_2)$. In addition we have to mod out the gauge transformations generated by $\tilde{m}^{i-}$. On top of the transformations in (D.33), this also includes

$$\tilde{m}_{\mathbf{1}}^{i-} = \begin{pmatrix} \operatorname{Re} K \wedge (\tilde{a} \operatorname{Re}\Phi_1 + \tilde{d} \operatorname{Im}\Phi_1) - \operatorname{Im} K \wedge (\tilde{b} \operatorname{Re}\Phi_1 + \tilde{c} \operatorname{Im}\Phi_1) \\ \operatorname{Re} K \wedge (\tilde{b} \operatorname{Re}\Phi_1 + \tilde{c} \operatorname{Im}\Phi_1) + \operatorname{Im} K \wedge (\tilde{a} \operatorname{Re}\Phi_1 + \tilde{d} \operatorname{Im}\Phi_1) \end{pmatrix} \qquad (\text{D.46})$$
$$= \operatorname{Re}(c_3 u^i K \wedge \bar{\Phi}_1 + c_4 u^i K \wedge \Phi_1) \,.$$

Together with the $\mathrm{SO}(4)$ rotations among $\operatorname{Re}\Phi_1$, $\operatorname{Im}\Phi_1$, $\operatorname{Re}\Phi_2$ and $\operatorname{Im}\Phi_2$ and the $\mathrm{SO}(2)$ rotation of u^i, this forms an $\mathrm{SO}(6)$ group which consists of elements that leave λ and μ invariant up to gauge degrees of freedom.

So far we discussed only the transformations in $\mathrm{SO}(6,6)$. The transformations in $Sl(2,\mathbb{R})_T$ stabilize μ, but only an $\mathrm{SO}(2)$ subgroup leaves λ invariant up to gaugings. Therefore, we end up with the parameter space

$$\mathcal{M}_{\lambda,\mu} = \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6) \times \mathrm{SO}(6)} \times \frac{Sl(2,\mathbb{R})_T}{\mathrm{SO}(2)} \,. \qquad (\text{D.47})$$

The first factor describes the choice of a space-like six-dimensional subspace in a space of signature $(6,6)$. Analogously to section D.2.2, we can interpret this six-dimensional subspace as being spanned by the real and imaginary parts of $u^i K$, Φ_1 and Φ_2. In fact, the $E_{7(7)}$ covariant formalism describes the procedure to descend from the flat superconformal

cone over (D.47) to the moduli space itself. First, one assumes that the real and imaginary part of $u^i K$, Φ_1 and Φ_2, which are space-like $SO(6,6)$ vectors, do not feel any constraints on their position in the vector space of signature $(6,6)$ apart from being space-like and linearly independent. This defines a $Gl(6)$ bundle over (D.47), which is an open set in $\bigoplus_{i=1}^{6} \mathbb{R}^{(6,6)}$. The purity conditions (D.45) and the compatibility condition (3.59) together with gauge fixing then project this flat cone to (D.47). This cone also survives the transition to a global moduli space by performing the Kaluza-Klein truncation, as done in section 3.6, where it becomes the $Gl(6)$ bundle over the first factor in (3.61) and is an open set in $\bigoplus_{i=1}^{6} \mathbb{R}^{(6,6+n)}$.

At the end of this section, we want to identify explicitly the scalar degrees of freedom coming from the RR-sector. For this we have to distinguish the deformations that are $SU(2) \times SU(2)$ singlets from those that are in the $(\mathbf{2},\mathbf{2})$ representation. The deformations in the $(\mathbf{2},\mathbf{2})$ representation are in the $\mathbf{6}$ representation of both $SU(2,2)_{\Phi_1}$ and $SU(2,2)_{\Phi_2}$. Therefore, they are the deformations which are displayed in (D.37) but are not of the form (D.20). This gives eight degrees of freedom, corresponding to all RR-form fields that are in the $(\mathbf{2},\mathbf{2})$ representation of $SU(2) \times SU(2)$. Their action on μ is given by (D.39), while λ stays invariant.

The situation for the singlets is a bit more involved, because some of the deformations we presented in (D.20) and (D.36) might just refer to gauge transformations which we mod out. Indeed, exactly the deformations displayed in (D.46) are pure gauge, and therefore, out of the eight deformations in (D.20), only those of the form

$$\tilde{n}_{\mathbf{1}}^{i-} = \begin{pmatrix} \operatorname{Re} K \wedge (\tilde{a} \operatorname{Re} \Phi_1 + \tilde{d} \operatorname{Im} \Phi_1) + \operatorname{Im} K \wedge (\tilde{b} \operatorname{Re} \Phi_1 + \tilde{c} \operatorname{Im} \Phi_1) \\ \operatorname{Re} K \wedge (\tilde{b} \operatorname{Re} \Phi_1 + \tilde{c} \operatorname{Im} \Phi_1) - \operatorname{Im} K \wedge (\tilde{a} \operatorname{Re} \Phi_1 + \tilde{d} \operatorname{Im} \Phi_1) \end{pmatrix}$$
$$= \operatorname{Re}(d_3 u^i \bar{K} \wedge \bar{\Phi}_1 + d_4 u^i \bar{K} \wedge \Phi_1) \tag{D.48}$$

are physical.[34] Thus, in the singlets we have eight physical degrees of freedom, which are parameterized by (D.36) and (D.48). These together with the degrees of freedom in the $(\mathbf{2},\mathbf{2})$ representation form exactly one spinor $C \in \Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6$ and represent the RR-scalars. This spinor C is composed out of the several RR-fields via the formal sum

$$C = C_1 + C_3 + C_5 \in \Lambda^{\mathrm{odd}} T_2^* Y_6 \wedge \Lambda^{\mathrm{even}} T_4^* Y_6 \,, \tag{D.49}$$

and is identified with the deformations in (D.37), (D.36) and (D.48) via the decomposition

$$\begin{aligned} C &= \operatorname{Re} K \wedge (\tilde{a} \operatorname{Re} \Phi_1 + \tilde{d} \operatorname{Im} \Phi_1 + a \operatorname{Re} \Phi_2 + d \operatorname{Im} \Phi_2 - \alpha) \\ &\quad + \operatorname{Im} K \wedge (\tilde{b} \operatorname{Re} \Phi_1 + \tilde{c} \operatorname{Im} \Phi_1 + b \operatorname{Re} \Phi_2 + c \operatorname{Im} \Phi_2 + \beta) \\ &= \operatorname{Re}(d_1 \bar{K} \wedge \bar{\Phi}_2 + d_2 \bar{K} \wedge \Phi_2 + d_3 \bar{K} \wedge \bar{\Phi}_1 + d_4 \bar{K} \wedge \Phi_1 - K \wedge \beta_{\mathbb{C}}^{(\mathbf{2},\mathbf{2})}) \,, \end{aligned} \tag{D.50}$$

where α and β are the real $SO(4,4)$ spinors that are in the $(\mathbf{2},\mathbf{2})$ representation of $SU(2) \times SU(2)$, and $\alpha_{\mathbb{C}}^{(\mathbf{2},\mathbf{2})}$ is a complex $SO(4,4)$ spinor in the same representation. There is a simple way to display the RR-fields as $E_{7(7)}$ transformations that is

$$\tilde{m}^{i-} = \operatorname{Re}(u^i(C - \mathrm{i} *_B C)) \,. \tag{D.51}$$

---

[34]Note that these transformations deform λ but leave μ invariant.

If exponentiated, these transformations correspond to shifts in the RR-fields which read

$$C_0 \to C_0 + C \; . \tag{D.52}$$

Note that transformations of the form (D.51) commute with each other up to gauge transformations and elements in the stabilizer of the $SU(2) \times SU(2)$ structure.

So far we did not discuss shifts in the complexified dilaton $B_6 + i\,e^{-\phi}$.[35] These are the $Sl(2, \mathbb{R})_S$ transformations

$$\tilde{m}^i{}_j = \begin{pmatrix} e^\phi & 0 \\ -B_6 & e^{-\phi} \end{pmatrix} \, , \tag{D.53}$$

which span $\frac{Sl(2,\mathbb{R})_S}{SO(2)} \subset \frac{SO(6,6)}{SO(6) \times SO(6)}$.

E $SU(2) \times SU(2)$ structures in five dimensions

In this appendix we study the missing case of backgrounds of the form (1.1) with a five-dimensional Minkowskian space-time ($d = 5$) times a five-dimensional compact manifold Y_5. As before we focus on the situation where these backgrounds preserve 16 supercharges which corresponds to $N = 2$ in five dimensions. More precisely the Lorentz group for these backgrounds is $SO(1, 4) \times SO(5)$. The ten-dimensional spinor representation decomposes accordingly as[36]

$$\mathbf{16} \to (\mathbf{4}, \mathbf{4}) \, , \tag{E.1}$$

where the first $\mathbf{4}$ denotes a spinor of $SO(1, 4)$ while the second one denotes the spinor representation of $SO(5)$. Preserving half of the supercharges amounts to choosing backgrounds which admit one or two globally defined spinors which corresponds to manifolds Y_5 with a reduced structure group $SU(2)$ or $SU(2) \times SU(2)$, respectively. We will start with a general analysis of the spectrum of type II supergravities in such backgrounds. We discuss first geometrical $SU(2)$ structure backgrounds, which are then generalized by the use of the pure spinor methods of generalized geometry and its analogues in exceptional generalized geometry. Note that a similar, independently performed analysis of exceptional generalized geometry in $d = 5$ has been presented in [20].

E.1 Field decomposition for $d = 5$

In this section we want to analyze the massless type II supergravity fields in terms of their representations under the five-dimensional Lorentz group and the $SU(2) \times SU(2)$ structure group and show analogously to section 2.1 how they assemble in $N = 2, d = 5$ multiplets, in the spirit of [13].

Again, we use the light-cone gauge where on-shell the fields form representations of $SO(3)$ instead of the whole $SO(1, 4)$ Lorentz group. Since we treat the case of $SU(2) \times SU(2)$ structure group, we therefore examine the decomposition of massless type II supergravity

---

[35]Here, B_6 denotes the dualized B field which is a scalar in four dimensions. Our notation refers to the democratic formulation and expresses the four-dimensional scalar in terms of the six-form B_6 which is dual to the ten-dimensional B field and fills out all internal directions.

[36]Note that there are no chiral spinor representations for $SO(1, 4)$ and $SO(5)$.

fields under the group $\mathrm{SO}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. For this, let us recall the decomposition of the two Majorana-Weyl representations $\mathbf{8}^s$ and $\mathbf{8}^c$ and the vector representation $\mathbf{8}^v$ under the breaking $\mathrm{SO}(8) \to \mathrm{SO}(3) \times \mathrm{SO}(5) \to \mathrm{SO}(3) \times \mathrm{SU}(2)$. We get

$$
\begin{aligned}
\mathbf{8}^s &\to \mathbf{4_2} \to 2\,\mathbf{1_{\frac{1}{2}}} \oplus \mathbf{2_{\frac{1}{2}}}\,, \\
\mathbf{8}^c &\to \mathbf{4_2} \to 2\,\mathbf{1_{\frac{1}{2}}} \oplus \mathbf{2_{\frac{1}{2}}}\,, \\
\mathbf{8}^v &\to \mathbf{1_1} \oplus \mathbf{5_0} \to \mathbf{1_1} \oplus \mathbf{1_0} \oplus 2\,\mathbf{2_0}\,,
\end{aligned}
\tag{E.2}
$$

where the subscript denotes the spin s under the $\mathrm{SO}(3)$ Lorentz group, i.e. the representation has dimension $(2s+1)$. We note that both Majorana-Weyl representations decompose in the same way under a $\mathrm{SU}(2)$ structure for $d = 5$. Therefore, we expect compactifications of type IIA and type IIB to give the same low energy effective action in this dimension.

In type IIA the massless fermionic degrees of freedom originate from the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^c)$ representation of $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R$, while in type IIB they form the $(\mathbf{8}^s, \mathbf{8}^v)$ and $(\mathbf{8}^v, \mathbf{8}^s)$ representation. However, since $\mathbf{8}^s$ and $\mathbf{8}^c$ decompose in the same way, we just investigate the former one. Under the decomposition $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R \to \mathrm{SO}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ they behave as

$$
\begin{aligned}
(\mathbf{8}^s, \mathbf{8}^v) &\to 2(\mathbf{1},\mathbf{1})_{\frac{3}{2}} \oplus 4(\mathbf{1},\mathbf{1})_{\frac{1}{2}} \oplus 4(\mathbf{1},\mathbf{2})_{\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1})_{\frac{3}{2}} \oplus 2(\mathbf{2},\mathbf{1})_{\frac{1}{2}} \oplus 2(\mathbf{2},\mathbf{2})_{\frac{1}{2}}\,, \\
(\mathbf{8}^v, \mathbf{8}^s),(\mathbf{8}^v, \mathbf{8}^c) &\to 2(\mathbf{1},\mathbf{1})_{\frac{3}{2}} \oplus 4(\mathbf{1},\mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2})_{\frac{3}{2}} \oplus 2(\mathbf{1},\mathbf{2})_{\frac{1}{2}} \oplus 4(\mathbf{2},\mathbf{1})_{\frac{1}{2}} \oplus 2(\mathbf{2},\mathbf{2})_{\frac{1}{2}}\,.
\end{aligned}
\tag{E.3}
$$

We see that half of the gravitinos come in the $(\mathbf{1},\mathbf{1})$ representation while the other half is in the doublet representations $(\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1})$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The latter ones again correspond to massive gravitino multiplets that must be projected out to end up with standard $N = 2, d = 5$ supergravity. After this projection, the fermionic components in the $(\mathbf{1},\mathbf{1})$ become part of the gravity multiplet and one vector multiplet, while the $(\mathbf{2},\mathbf{2})$ components correspond to the fermionic degrees of freedom in the vector multiplets.[37]

The massless bosonic fields of type II supergravity can be decomposed in the same way. For the NS-NS-sector we consider the combination $E_{MN} = g_{MN} + B_{MN} + \phi \eta_{MN}$ which forms the $(\mathbf{8}^v, \mathbf{8}^v)$ representation. It decomposes as

$$
\begin{aligned}
E_{\mu\nu} &: (\mathbf{1},\mathbf{1})_{\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{1}}\,, \\
E_{\mu m} &: (\mathbf{1},\mathbf{1})_{\mathbf{1}} \oplus 2(\mathbf{1},\mathbf{2})_{\mathbf{1}}\,, \\
E_{m\mu} &: (\mathbf{1},\mathbf{1})_{\mathbf{1}} \oplus 2(\mathbf{2},\mathbf{1})_{\mathbf{1}}\,, \\
E_{mn} &: (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus 2(\mathbf{1},\mathbf{2})_{\mathbf{0}} \oplus 2(\mathbf{2},\mathbf{1})_{\mathbf{0}} \oplus 4(\mathbf{2},\mathbf{2})_{\mathbf{0}}\,,
\end{aligned}
\tag{E.4}
$$

where in the first line the first component corresponds to the metric, the second one to the five-dimensional dilaton and the third one to the antisymmetric tensor field. After the projection we are not only left with the four-dimensional metric and the antisymmetric two-tensor but also with two vectors of the form $E_{\mu m}$ and $E_{m\mu}$ and one scalar $E_{mn}^{(\mathbf{1},\mathbf{1})}$

---

[37]As we know from section 2.1, the $(\mathbf{1},\mathbf{1})$ representation corresponds to the gravity multiplet in $d = 6$ (for type IIA). This decomposes into the gravity multiplet plus one vector multiplet in the $(\mathbf{1},\mathbf{1})$ representation in $d = 5$.

that become part of the gravity multiplet and the vector multiplet which was already mentioned. The other scalars $E_{mn}^{(2,2)}$ sit in a vector multiplet. As in the $d = 6$ case, they can be associated with deformations of the $SU(2) \times SU(2)$ background.

Finally, we decompose the RR-sector. This corresponds to the decomposition of the $(\mathbf{8}^s, \mathbf{8}^c)$ representation or the $(\mathbf{8}^s, \mathbf{8}^s)$, leading here to the same result. We find

$$
\begin{aligned}
(\mathbf{8}^s, \mathbf{8}^c), (\mathbf{8}^s, \mathbf{8}^s) \rightarrow &4(\mathbf{1}, \mathbf{1})_{\mathbf{1}} \oplus 4(\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus 2(\mathbf{1}, \mathbf{2})_{\mathbf{1}} \oplus 2(\mathbf{1}, \mathbf{2})_{\mathbf{0}} \\
&\oplus 2(\mathbf{2}, \mathbf{1})_{\mathbf{1}} \oplus 2(\mathbf{2}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{2}, \mathbf{2})_{\mathbf{1}} \oplus (\mathbf{2}, \mathbf{2})_{\mathbf{0}} \ .
\end{aligned}
\tag{E.5}
$$

Projecting out all $SU(2) \times SU(2)$ doublets leaves us with four vectors and four scalars in the $(\mathbf{1}, \mathbf{1})$ representation and one vector and one scalar in the $(\mathbf{2}, \mathbf{2})$ representation.

These fields can be assembled into a gravity multiplet and two vector multiplets. The gravity multiplet contains the graviton, four gravitini, six vector fields, four Weyl fermions and a real scalar all in the $(\mathbf{1}, \mathbf{1})$ representation. The two vector multiplets are in the $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{2})$ representation respectively, and each contains one vector field, four gaugini and five scalars.

Analogously to section 2.1 we still deal with ten-dimensional fields which have been reordered in such a way that they form $N = 2, d = 5$ multiplets. The action corresponding to these multiplets only allows for manifest $SO(1, 4) \times SO(5)$ symmetry and $N = 2$ super-symmetry. Then we projected out the $SU(2) \times SU(2)$ doublets to achieve an action that is only $N = 2$ supersymmetric.

E.2 $SU(2)$ structures on manifolds of dimension 5

For a five-dimensional manifold to admit a globally defined, nowhere vanishing $SO(5)$ spinor η requires the structure group G to be contained in $SU(2)$ since this is the largest subgroup that allows for a singlet in the spinor representation of $SO(5)$. This can be seen by the fact that the spin double cover of $SO(5)$ is $Sp(2)$. To allow for a spinor singlet η, this group has to be broken to $Sp(1) \equiv SU(2)$. We will assume in the following that η is normalized to one. Considering the breaking $SO(5) \rightarrow SU(2)$, the spinor representation splits like

$$
\mathbf{4} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \ .
\tag{E.6}
$$

Here, η and η^c define the two singlets. From these two singlets, one can globally define the nowhere vanishing two-forms J and Ω using (2.9). Furthermore, one can define a real one-form L, given by

$$
L_m = \bar{\eta}\gamma_m\eta \, ,
\tag{E.7}
$$

which defines an almost product structure

$$
P^m{}_n = 2L^m L_n - \delta^m{}_n \ .
\tag{E.8}
$$

It singles out one direction L in TY_5 at each point and breaks the structure group to $SO(4)$. Similarly to the discussion of appendix C, all vectors orthogonal to L are $SU(2)$ doublets. Therefore, in the $N = 2$ theory only the prefactor of L can be deformed and P is rigid. Analogously to the four-dimensional case, one can define two-forms via (2.9) which reduce

the structure group further to SU(2). For the SU(2) structure part of the geometric moduli space, the discussion of section 2.2 applies. We computed the geometric moduli space to be (2.53). Here, the prefactor of L gives one additional degree of freedom. Thus, the result is

$$\mathcal{M}_{d=5}^{\text{geom}} = \frac{\text{SO}(3, n+3)}{\text{SO}(3) \times \text{SO}(n+3)} \times \mathbb{R}_+ \times \mathbb{R}_+ \ . \tag{E.9}$$

E.3 Pure spinors in five dimensions

Now we want to apply the framework of generalized geometry to the case $d = 5$. However, one cannot define a valid generalized almost complex structure on the ten-dimensional bundle $TY_5 \oplus T^*Y_5$ compatible with the canonical pairing $\mathcal{I}$. This generalizes the fact that one cannot define an almost complex structure on a manifold of odd dimension. Therefore, the techniques of generalized geometry seem not to apply for a manifold of dimension five. However, the language of pure spinors still makes sense, and therefore we can apply it even in this case.

In five dimensions, an SO(5,5) spinor is defined to be pure if and only if (see for example [48])

$$\langle \Phi, \Gamma^M \Phi \rangle_5 = 0 \qquad \text{for all} \quad M = 1, \dots, 10 \ . \tag{E.10}$$

Furthermore, we define a pure SO(5,5) spinor to be normalizable if and only if

$$\langle \Phi, \Gamma^M \bar{\Phi} \rangle_5 \neq 0 \ . \tag{E.11}$$

For two pure spinors Φ^+ and Φ^- of opposite chirality, we have to define appropriate compatibility conditions. Since we are not able to switch to the language of generalized almost complex structures, we cannot use the intuition which is usually gained there. However, in analogy to the case of a manifold of dimension four, we can define the compatibility conditions

$$\langle \Phi^+, \Phi^- \rangle_5 = \langle \Phi^+, \bar{\Phi}^- \rangle_5 = 0 \tag{E.12}$$

and the normalization

$$\langle \Phi^+, \Gamma^M \bar{\Phi}^+ \rangle_5 \, \mathcal{I}_{MN} \, \langle \Phi^-, \Gamma^N \bar{\Phi}^- \rangle_5 = c \,, \tag{E.13}$$

where $\mathcal{I}_{MN}$ is the SO(5,5) metric induced by the natural pairing of tangent and cotangent space. Here it becomes clear why we chose $\Phi^\pm$ to be of opposite chirality. Only for such pairs, eq. (E.13) can be different from zero. We choose in the following the gauge $c = 1$.

A pair $\Phi^\pm$ with these properties already defines some generalized almost product structure $\mathcal{P}$ via

$$\mathcal{P}^M{}_N = \left(\langle \Phi^+, \Gamma^M \bar{\Phi}^+ \rangle_5 \langle \Phi^-, \Gamma^K \bar{\Phi}^- \rangle_5 + \langle \Phi^+, \Gamma^K \bar{\Phi}^+ \rangle_5 \langle \Phi^-, \Gamma^M \bar{\Phi}^- \rangle_5 \right) \mathcal{I}_{KN} - \delta^M{}_N \ . \tag{E.14}$$

Since the pure spinors are globally defined, $\mathcal{P}$ is globally defined. It is symmetric with respect to $\mathcal{I}$ and divides $TY_5 \oplus T^*Y_5$ into some two-dimensional eigenspace with eigenvalue $+1$ and an eight-dimensional eigenspace with eigenvalue -1. This splitting is compatible with $\mathcal{I}$ and therefore breaks SO(5,5) to SO(4,4) $\times$ SO(1,1). Note that in contrast to the

case $d = 4$, where we had to impose the existence of $\mathcal{P}$ in addition to the existence of a pure spinor pair, here it just results from the definition of $\Phi^\pm$.

We can use this splitting to decompose Φ^+ and Φ^- into the corresponding pure SO(4, 4) spinors and the SO(1, 1) spinor parts. Projecting out all $SU(2) \times SU(2)$ doublets implies that the SO(4, 4) spinor components must be of even degree. Thus, we find

$$\Phi^+ = 1 \wedge \Phi_1, \qquad \Phi^- = L \wedge \Phi_2, \tag{E.15}$$

where L is the one-form that is in the $+1$ eigenspace of $\mathcal{P}$. Here, Φ_1 and Φ_2 are compatible pure SO(4, 4) spinors of even chirality that define some $SU(2) \times SU(2)$ structure, while L is the one-form that corresponds to the fifth direction. Note that the ratio of volumes of the SO(4, 4) and the $SO(1, 1)$ direction can be reparameterized by the SO(1, 1) vector

$$U^K = \langle \Phi^+, \Gamma^K \bar{\Phi}^+ \rangle_5 + \langle \Phi^-, \Gamma^K \bar{\Phi}^- \rangle_5. \tag{E.16}$$

Now we can again do a Kaluza-Klein truncation to obtain a finite-dimensional moduli space. By using the same methods as in section 2.4 we can determine the moduli space of $SU(2) \times SU(2)$ structures. On top of (2.51) we get one additional $\mathbb{R}_+$ factor parameterized by U^K, corresponding to the length of L. Therefore, the moduli space is

$$\mathcal{M}^{\mathrm{NS}}_{d=5} = \frac{\mathrm{SO}(4, n + 4)}{\mathrm{SO}(4) \times \mathrm{SO}(n + 4)} \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{E.17}$$

E.4 Exceptional generalized geometry in five dimensions

Analogously to $d = 6$ we can use the U-duality covariant formalism to include the RR scalars into the moduli space. The U-duality group in five dimensions is $E_{6(6)}$ with the T-duality subgroup being SO(5, 5) (for more details on the group $E_{6(6)}$ see [49]). We discuss now the decomposition of the representations of $E_{6(6)}$ in terms of the maximal subgroup $\mathbb{R}_+ \times \mathrm{SO}(5, 5)$, where the $\mathbb{R}_+$ factor refers to shifts in the dilaton ϕ. The fundamental representation of $E_{6(6)}$ is of complex dimension 27 and decomposes as

$$\mathbf{27} \to \mathbf{1}_{+4} + \mathbf{10}_{-2} + \mathbf{16}_{+1}. \tag{E.18}$$

$E_{6(6)}$ has also an anti-fundamental representation which decomposes analogously as

$$\bar{\mathbf{27}} \to \mathbf{1}_{-4} + \mathbf{10}_{+2} + \bar{\mathbf{16}}_{-1}. \tag{E.19}$$

The adjoint of $E_{6(6)}$ decomposes as

$$\mathbf{78} \to \mathbf{1}_0 + \mathbf{16}_{-3} + \bar{\mathbf{16}}_{+3} + \mathbf{45}_0. \tag{E.20}$$

To find the geometric realizations of these representations, we consider the charges in $d = 5$, which form the fundamental representation of $E_{6(6)}$, analogously to section 3.5. One finds [18, 43]

$$\mathbf{27}^{\mathrm{IIA}} \to (\mathbb{R})_{+4} + (TY_5 \oplus T^*Y_5)_{-2} + (\Lambda^{\mathrm{even}} T^*Y_5)_{+1} \tag{E.21}$$

for type IIA, and

$$\mathbf{27}^{\mathrm{IIB}} \to (\mathbb{R})_{+4} + (TY_5 \oplus T^*Y_5)_{-2} + (\Lambda^{\mathrm{odd}} T^*Y_5)_{+1} \tag{E.22}$$

for type IIB. From (E.21) we can derive the geometrical realization of the adjoint given in (E.20) for type IIA, which is

$$e_{6(6)}^{\mathrm{IIA}} = (\mathbb{R})_0 \oplus (\Lambda^{\mathrm{odd}}T^*Y_5)_{-3} \oplus (\Lambda^{\mathrm{even}}T^*Y_5)_{+3} \oplus (so(TY_5 \oplus T^*Y_5))_0 \ . \tag{E.23}$$

For type IIB, only the charges of the dilaton are inverted. Therefore, we expect that both theories have the same moduli space.[38] In (E.23) the spinor representations of SO(5,5) correspond to the additional generators of the group $E_{6(6)}$.

Following (E.21), we decompose an element λ in the fundamental representation of $E_{6(6)}$ into its components

$$\lambda = (\lambda_0, \lambda^A, \lambda^+), \qquad A = 1, \dots, 10, \tag{E.24}$$

and similarly elements ρ in the anti-fundamental representation

$$\rho = (\rho_0, \rho^A, \rho^-) \ . \tag{E.25}$$

In the same way, we can write an element M in the adjoint representation as

$$M = (m_0, m^+, m^-, m^A{}_B) \ . \tag{E.26}$$

The action of the adjoint representation on the fundamental is given by

$$\begin{aligned} M \cdot \lambda = &(4m_0\lambda_0 + \langle m^-, \lambda^+ \rangle_5, \ m^A{}_B\lambda^B - 2m_0\lambda^A + \langle m^+, \Gamma^A\lambda^+ \rangle_5, \\ &m_{AB}\Gamma^{AB}\lambda^+ + \lambda^A\mathcal{I}_{AB}\Gamma^B m^- + m_0\lambda^+ + \lambda_0 m^+) \ . \end{aligned} \tag{E.27}$$

The product of two **27**s contains a $\bar{\mathbf{27}}$ which is related to the existence of the cubic invariant of $E_{6(6)}$. More precisely, we have $\mathbf{27} \times \mathbf{27} \to \bar{\mathbf{27}} \oplus \dots$ which by projection leads to the map

$$\begin{aligned} \mathbf{27} \times \mathbf{27} &\longrightarrow \bar{\mathbf{27}}, \\ (\lambda, \lambda') &\longmapsto \lambda \times \lambda', \end{aligned} \tag{E.28}$$

where

$$\lambda \times \lambda' = (\lambda^A\mathcal{I}_{AB}\lambda'^B, \ \lambda_0\lambda'^A + \lambda'_0\lambda^A + \langle \lambda^+, \Gamma^A\lambda'^+ \rangle_5, \ \lambda^A\mathcal{I}_{AB}\Gamma^B\lambda'^+ + \lambda'^A\mathcal{I}_{AB}\Gamma^B\lambda^+) \ . \tag{E.29}$$

The scalar product

$$\begin{aligned} \mathbf{27} \times \bar{\mathbf{27}} &\longrightarrow \mathbf{1}, \\ (\lambda, \rho) &\longmapsto \lambda \cdot \rho, \end{aligned} \tag{E.30}$$

is defined as

$$\lambda \cdot \rho = \lambda_0\rho_0 + \lambda^A\mathcal{I}_{AB}\rho^B + \langle \lambda^+, \rho^- \rangle_5 \ . \tag{E.31}$$

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[38] Actually, we just have to interchange $\Lambda^{\mathrm{odd}}T^*Y_5$ and $\Lambda^{\mathrm{even}}T^*Y_5$ everywhere to get the result for type IIB. In terms of the pure spinors in (E.15), we have to apply the operator $\tau_L \equiv \iota_L + L\wedge \in TY_5 \oplus T^*Y_5$ which changes the chirality of Φ^+ and Φ^-. This symmetry is just the generalization of T-duality. Up to the gauging that interchanges Φ_1 and Φ_2 in (E.15), this just corresponds to the exchange of the two pure spinors Φ^+ and Φ^-, which is mirror symmetry.

Next we embed the pair of compatible pure $SO(5,5)$ spinors Φ^+ and Φ^- into $E_{6(6)}$ representations. It seems most natural to embed them into the complexified fundamental and anti-fundamental representation, respectively. Therefore, we define

$$\lambda = (0,0,\Phi^+)\,, \qquad \rho = (0,0,\Phi^-)\,. \tag{E.32}$$

Then the purity condition (E.10) is easily rephrased in

$$\lambda \times \lambda = 0\,, \qquad \rho \times \rho = 0\,, \tag{E.33}$$

and we can impose the compatibility conditions (E.12) in the form

$$\lambda \cdot \rho = \bar{\lambda} \cdot \rho = 0\,. \tag{E.34}$$

The normalization condition (E.13) reads

$$(\lambda \times \bar{\lambda}) \cdot (\rho \times \bar{\rho}) = 1\,. \tag{E.35}$$

Note that λ and ρ define some isomorphism between the $\mathbf{27}$ and the $\bar{\mathbf{27}}$ representation, via

$$\begin{aligned} I\ \ &\mathbf{27} \longrightarrow \bar{\mathbf{27}}\,, \\ &\tilde{\lambda} \longmapsto (\rho \times \bar{\rho}) \times \tilde{\lambda}\,, \end{aligned} \tag{E.36}$$

and

$$\begin{aligned} I^{-1}\ \ &\bar{\mathbf{27}} \longrightarrow \mathbf{27}\,, \\ &\tilde{\rho} \longmapsto (\lambda \times \bar{\lambda}) \times \tilde{\rho}\,, \end{aligned} \tag{E.37}$$

where I^{-1} is the inverse of I due to (E.35).

However, so far we did not embed the dilaton degree of freedom into some $E_{6(6)}$ representation. Analogously to (2.63), we can embed some $SO(1,1)$ vector into the fundamental representation of $E_{6(6)}$ which reflects the dilaton degree of freedom. First of all, one can either embed this doublet into the fundamental or into the anti-fundamental. However, both are related by the isomorphism (E.36). Therefore, it does not make any difference whether we use the fundamental or the anti-fundamental representation. The canonical $SO(1,1)$ doublet should correspond to the two singlets in (E.18) and (E.19) because these do not transform under any geometric structure group and are of opposite charge. These two singlets are not part of the same representation, but we can use (E.37) to map one of the singlets into the other representation. Therefore, the embedded $SO(1,1)$ doublet is spanned by the two elements in the $\mathbf{27}$ that are

$$\mu^1 = (1,0,0)_{\mathbf{27}}\,, \tag{E.38}$$

and

$$\mu^2 = (\lambda \times \bar{\lambda}) \times (1,0,0)_{\bar{\mathbf{27}}}\,. \tag{E.39}$$

The embedding of the dilaton into the fundamental representation of $E_{6(6)}$ then reads

$$\phi \longmapsto \mu(\phi) = \mathrm{e}^\phi \mu^1 + \mathrm{e}^{-\phi} \mu^2\,. \tag{E.40}$$

We choose the normalization

$$\mu \times \mu = \frac{1}{2} \, \lambda \times \bar{\lambda} \,. \tag{E.41}$$

We can always switch between μ and its 'cousin' $\hat{\mu}$ in the anti-fundamental by

$$\hat{\mu} = (\rho \times \bar{\rho}) \times \mu \,, \tag{E.42}$$

$$\mu = (\lambda \times \bar{\lambda}) \times \hat{\mu} \,. \tag{E.43}$$

Equation (E.41) together with (E.35) then just states that

$$\mu \cdot \hat{\mu} = 1 \,. \tag{E.44}$$

Finally, we impose the conditions

$$\mu \times \lambda = \mu \times \bar{\lambda} = 0 \tag{E.45}$$

and

$$\hat{\mu} \times \rho = \hat{\mu} \times \bar{\rho} = 0 \tag{E.46}$$

to make μ compatible with λ and ρ.

Now we want to discuss the effect of the projection to $N = 2$ which eliminates all $SU(2) \times SU(2)$ doublets. We have seen in section E.3 that this fixes the generalized almost product structure $\mathcal{P}$, defined in (E.14), and therefore reduces the T-duality group $SO(5,5)$ to $SO(4,4) \times \mathbb{R}_+$. Furthermore we know that modding out $SU(2) \times SU(2)$ doublets eliminates half of the degrees of freedom in $\Lambda^{\mathrm{odd}} T^* Y_5$ and $\Lambda^{\mathrm{even}} T^* Y_5$, while for $SO(5,5)$ vectors, only the components in the $+1$ eigenspace of $\mathcal{P}$ survive the projection. We decompose (E.20) further so that the adjoint in terms of $SO(4,4) \times \mathbb{R}_+ \times \mathbb{R}_+$ representations reads

$$\mathbf{78} \to \mathbf{1}_{0,0} + \mathbf{8}^f_{-3,+1} + \mathbf{8}^c_{-3,-1} + \mathbf{8}^f_{+3,-1} + \mathbf{8}^c_{+3,+1} + \mathbf{1}_{0,0} + \mathbf{8}^v_{0,+2} + \mathbf{8}^v_{0,-2} + \mathbf{28}_{0,0} \,. \tag{E.47}$$

Here, we project out the representations $\mathbf{8}^v$ and $\mathbf{8}^c$ because they consist of $SU(2) \times SU(2)$ doublets. Using the splitting of the adjoint of $SO(5,5)$ into $SO(4,4)$ representations, i.e.

$$\mathbf{45} \to \mathbf{28}_0 + \mathbf{8}_{+2} + \mathbf{8}_{-2} + \mathbf{1}_0 \,, \tag{E.48}$$

we identify the surviving pieces of (E.47) with the Lie algebra of $SO(5,5) \times \mathbb{R}_+$. Here the extra $\mathbb{R}_+$ factor is a combination of the volume of the fifth direction and the dilaton degree of freedom. More precisely, to match the representations, we have to transform the charges under the two Abelian $\mathbb{R}_+$ factors as

$$(p,q) \to \left(\frac{p-q}{2}, -\frac{p+3q}{2} \right) \,. \tag{E.49}$$

Next we consider the decomposition of the fundamental representation of $E_{6(6)}$ in terms of $SO(4,4) \times \mathbb{R}_+ \times \mathbb{R}_+$ representations

$$\mathbf{27} \to \mathbf{1}_{+4,0} + \mathbf{8}^v_{-2,0} + \mathbf{1}_{-2,+2} + \mathbf{1}_{-2,-2} + \mathbf{8}^f_{+1,+1} + \mathbf{8}^c_{+1,-1} \,. \tag{E.50}$$

Again, the representations $\mathbf{8}^v$ and $\mathbf{8}^c$ are projected out because they consist of $\mathrm{SU}(2){\times}\mathrm{SU}(2)$ doublets. The surviving pieces form an $\mathrm{SO}(5,5) \times \mathbb{R}_+$ vector

$$\mathbf{10}_{-2} \to \mathbf{8}^f_{0,-2} + \mathbf{1}_{+2,-2} + \mathbf{1}_{-2,-2} \tag{E.51}$$

and a singlet $\mathbf{1}_{+4}$. The same holds for the anti-fundamental representation, with the only difference that the Abelian charges have the opposite sign. The two singlets are spanned by $m_0 = (\bar{\lambda} \times \lambda)$ and $n_0 = (\bar{\rho} \times \rho)$, and both (E.29) and (E.31) become the usual scalar product between $\mathrm{SO}(5,5)$ vectors. λ, ρ and μ are projected to vector representations of $\mathrm{SO}(5,5)$, and $(\mathrm{Re}\,\lambda)$, $(\mathrm{Im}\,\lambda)$, $(n_0\,\mathrm{Re}\,\rho)$, $(n_0\,\mathrm{Im}\,\rho)$ and μ form a set of five orthonormal space-like $\mathrm{SO}(5,5)$ vectors. Therefore, they span the coset $\mathrm{SO}(5,5)/\mathrm{SO}(5)$. From now on everything works analogously to section 2.5. After modding out the symmetry between these five vectors, we end up with

$$\mathcal{M}_{\lambda,\rho,\mu} = \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5) \times \mathrm{SO}(5)} \times \mathbb{R}_+ \,, \tag{E.52}$$

where the $\mathbb{R}_+$ factor is due to m_0 and n_0, which form an $\mathrm{SO}(1,1)$ doublet parameterizing this degree of freedom. Using again the general argument that we presented in section 2.4, one can argue that after Kaluza-Klein truncation the actual moduli space is of the form

$$\mathcal{M}_{d=5} = \frac{\mathrm{SO}(5,n+5)}{\mathrm{SO}(5) \times \mathrm{SO}(n+5)} \times \mathbb{R}_+ \,. \tag{E.53}$$

The action of the RR fields can be determined analogously to section 2.5, with the same result as (2.66).

As we already discussed above, after the projection to a theory with 16 supercharges the real and imaginary parts of λ and ρ, which were introduced in (E.32), and $\mu(\phi)$ given in (E.40) form a set of five space-like, linearly independent vectors in $\mathbb{R}^{(5,n+5)}$. As long as we do not constrain these objects by purity or compatibility conditions, they parameterize the flat cone over (E.53) which is the moduli space of the corresponding superconformal supergravity. Imposing the purity and compatibility conditions given in (E.33), (E.34), (E.35), (E.41), (E.45) and (E.46) and the removal of gauge degrees of freedom reduce this to the moduli space (E.53).

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